Introduction to Bayesian Inference and Uncertainty Propagation

Ralph C. Smith
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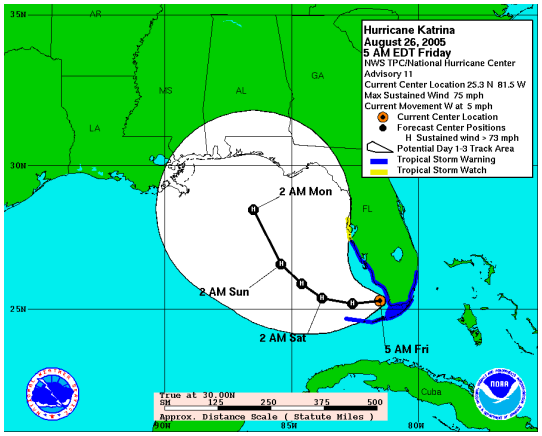

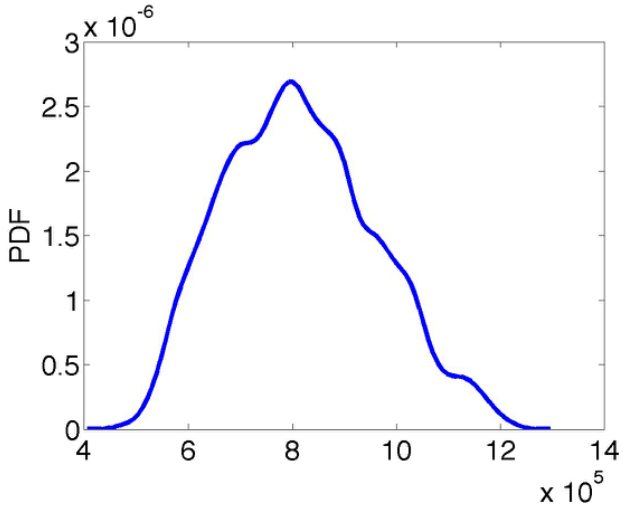

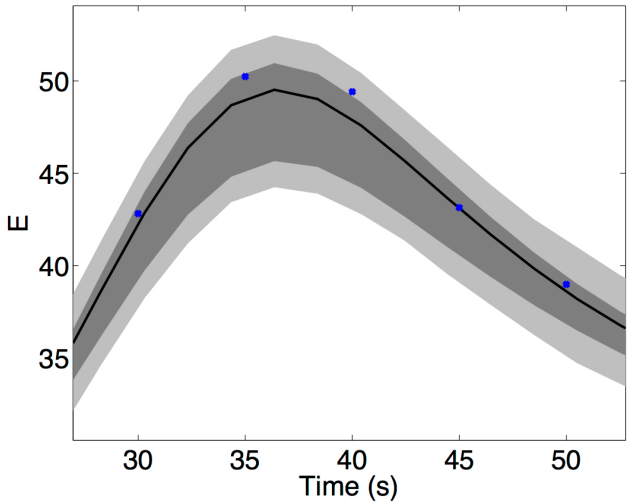

Essentially, all models are wrong, but some are useful, George E.P. Box, Industrial Statistician.

Support: DOE Consortium for Advanced Simulation of LWR (CASL)
NNSA Consortium for Nonproliferation Enabling Capabilities (CNEC)
National Science Foundation (NSF)
Air Force Office of Scientific Research (AFOSR)

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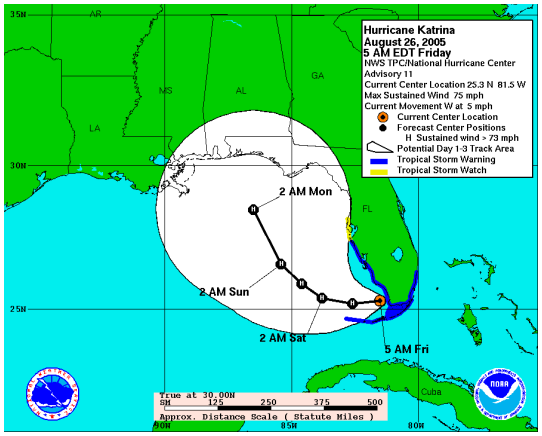
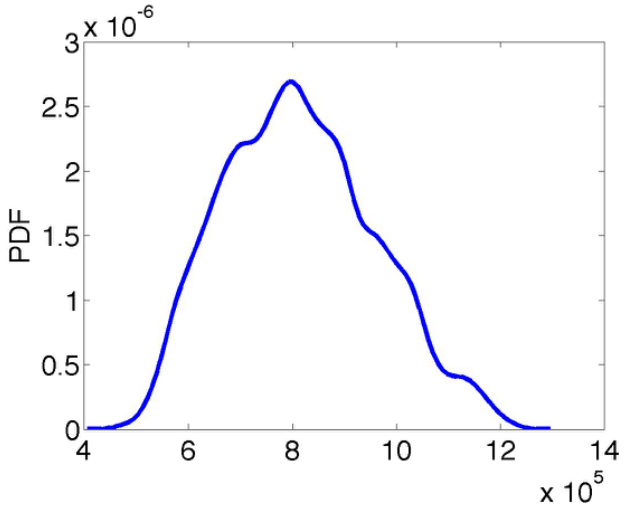
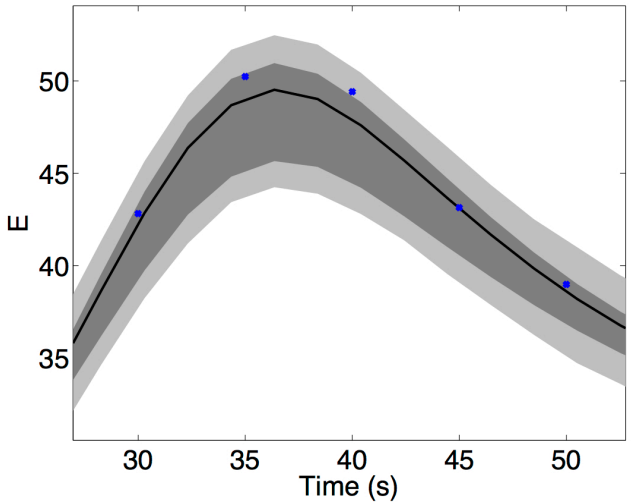






"We":

Zhengzheng Hu, Nate Burch, Allison Lewis, Kathleen Schmidt, Nikolas Bravo, Mami Wentworth (NCSU)

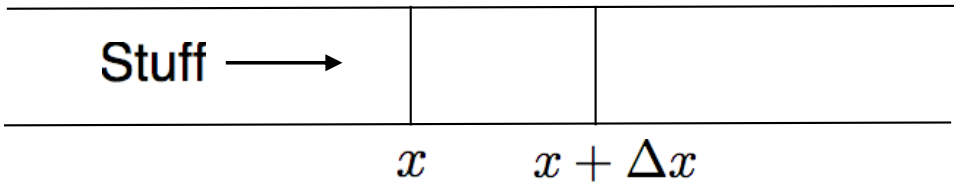
Michael Hays, Billy Oates (Florida State University)

Brian Williams (LANL), Russell Hooper, Brian Adams, Vince Mousseau (Sandia)

Emre Tatli and Yixing Sung (Westinghouse)

Modeling Strategy

General Strategy: Conservation of stuff

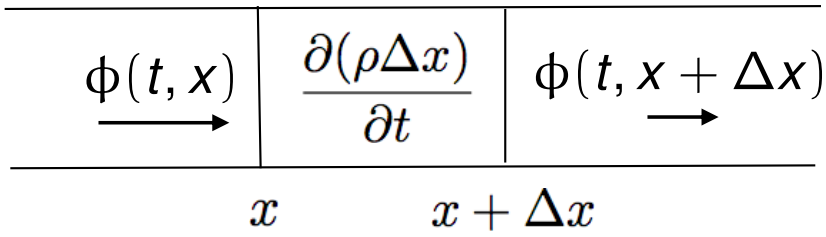


$$\frac{dStuff}{dt} = \text{Stuff in - Stuff out + Stuff created - Stuff destroyed}$$

Continuity Equation:

$$\frac{\partial(\rho\Delta x)}{\partial t} = \phi(t,x) - \phi(t,x+\Delta x)$$

$$\Rightarrow \lim_{\Delta x \to 0} \frac{\partial \rho}{\partial t} = \lim_{\Delta x \to 0} \frac{\phi(t,x) - \phi(t,x+\Delta x)}{\Delta x}$$



$$\Rightarrow \frac{\partial \rho}{\partial t} + \frac{\partial \phi}{\partial x} = 0$$

Density: $\rho(t,x)$ - Stuff per unit length or volume

Rate of Flow: $\phi(t,x)$ - Stuff per second

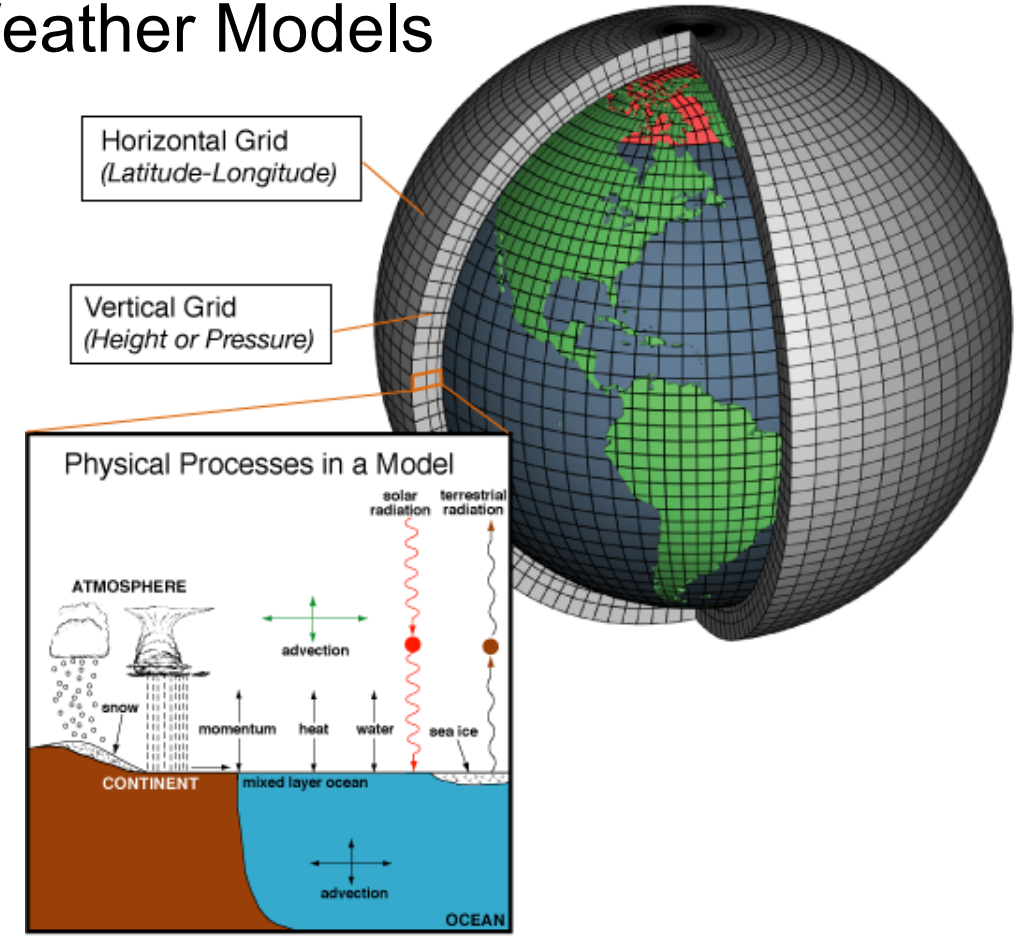
More Generally:

$$\Rightarrow \frac{\partial \rho}{\partial t} + \frac{\partial \phi}{\partial x} = \text{Sources - Sinks}$$

Example 1: Weather Models

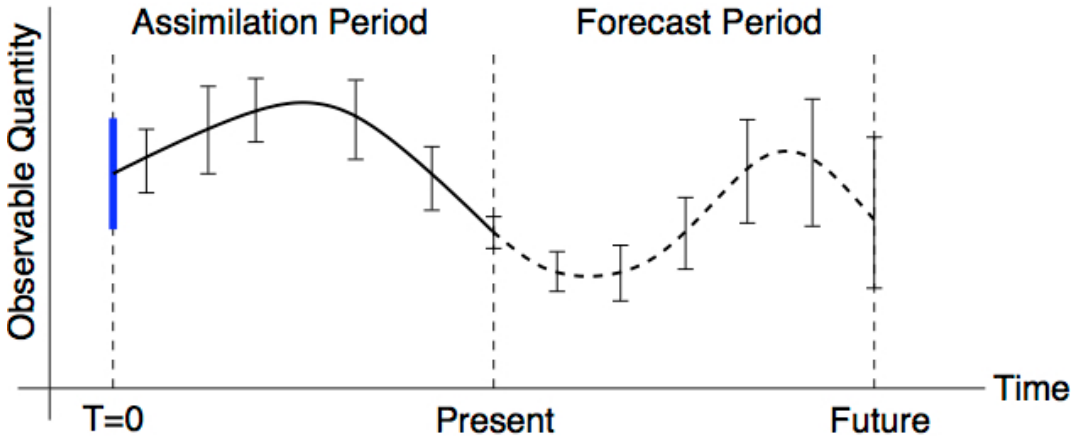
Challenges:

- Coupling between temperature, pressure gradients, precipitation, aerosol, etc.;
- Models and inputs contain uncertainties;
- Numerical grids necessarily larger than many phenomena; e.g., clouds
- Sensors positions may be uncertain; e.g., weather balloons, ocean buoys.



Goal:

- Assimilate data to quantify uncertain initial conditions and parameters;
- Make predictions with quantified uncertainties.



Equations of Atmospheric Physics

Conservation Relations:

$$\frac{\partial \rho}{\partial t} + \frac{\partial \phi}{\partial x} = \text{Sources - Sinks}$$

Mass $$\frac{\partial \rho}{\partial t} + \nabla \cdot (\rho v) = 0$$

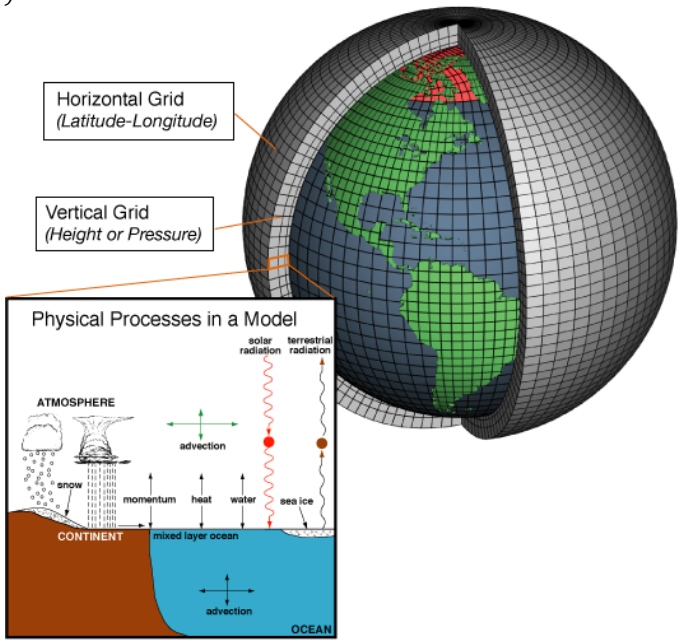
Momentum $$\frac{\partial v}{\partial t} = -v \cdot \nabla v - \frac{1}{\rho}\nabla p - g\hat{k} - 2\Omega \times v$$

Energy $$\rho c_V \frac{\partial T}{\partial t} + p\nabla \cdot v = -\nabla \cdot F + \nabla \cdot (k\nabla T) + \rho \dot{q}(T, p, \rho)$$

$$p = \rho RT$$

Water $$\frac{\partial m_j}{\partial t} = -v \cdot \nabla m_j + S_{m_j}(T, m_j, \chi_j, \rho)\ ,\ j = 1, 2, 3,$$

Aerosol $$\frac{\partial \chi_j}{\partial t} = -v \cdot \nabla \chi_j + S_{\chi_j}(T, \chi_j, \rho)\ ,\ j = 1, \cdots, J,$$



Constitutive Closure Relations: e.g.,

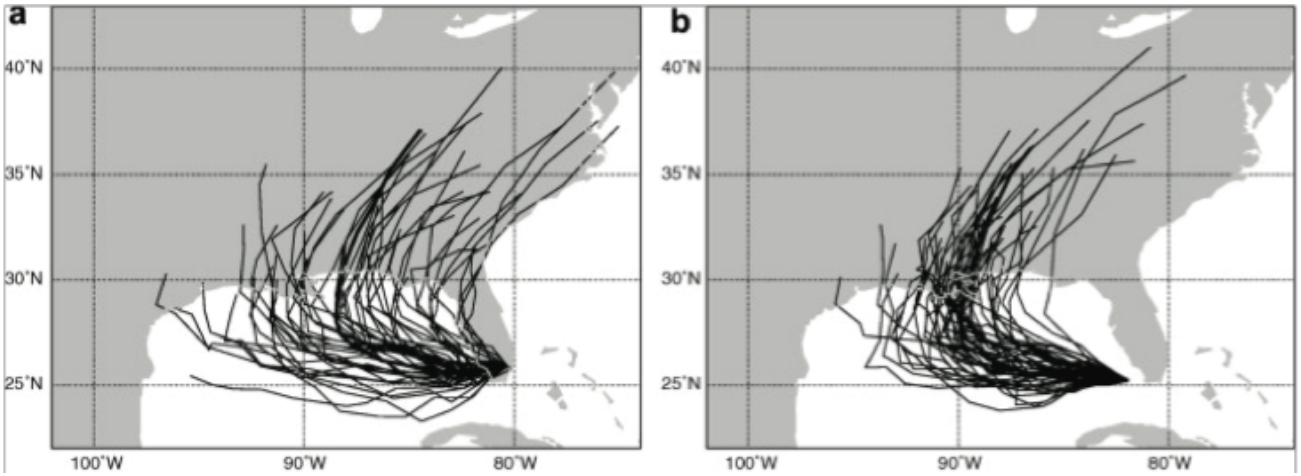
$$S_{m_2} = S_1 + S_2 + S_3 - S_4$$

where

$$S_1 = \bar{\rho}(m_2 - m_2^*)^2 \left[1.2 \times 10^{-4} + \left(1.569 \times 10^{-12} \frac{n_r}{d_0(m_2 - m_2^*)}\right)\right]^{-1}$$

Ensemble Predictions

Ensemble Predictions:



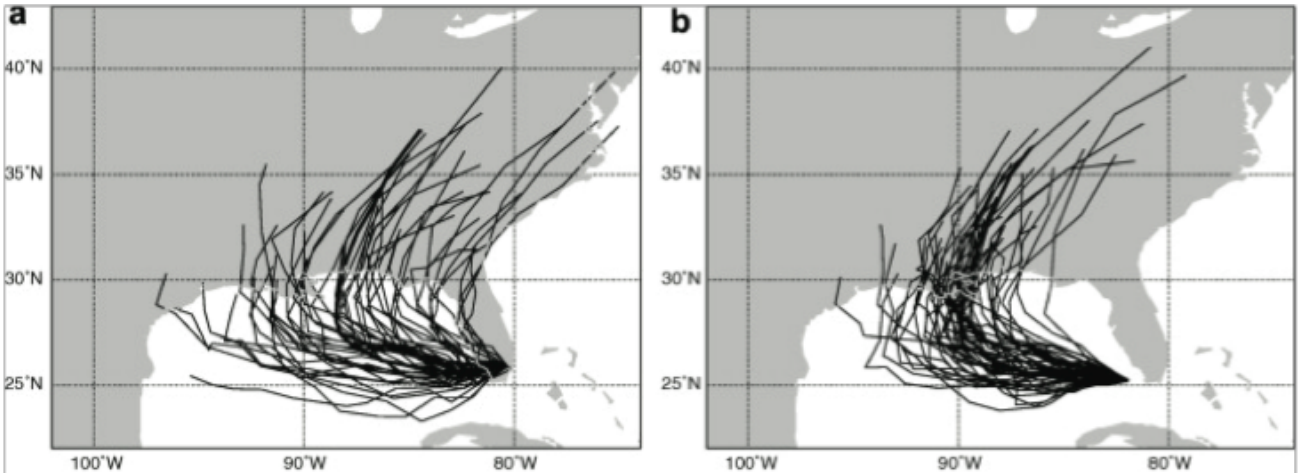
00 UTC on August 26, 2005

12 UTC on August 26, 2005

Cone of Uncertainty:

Ensemble Predictions

Ensemble Predictions:



00 UTC on August 26, 2005

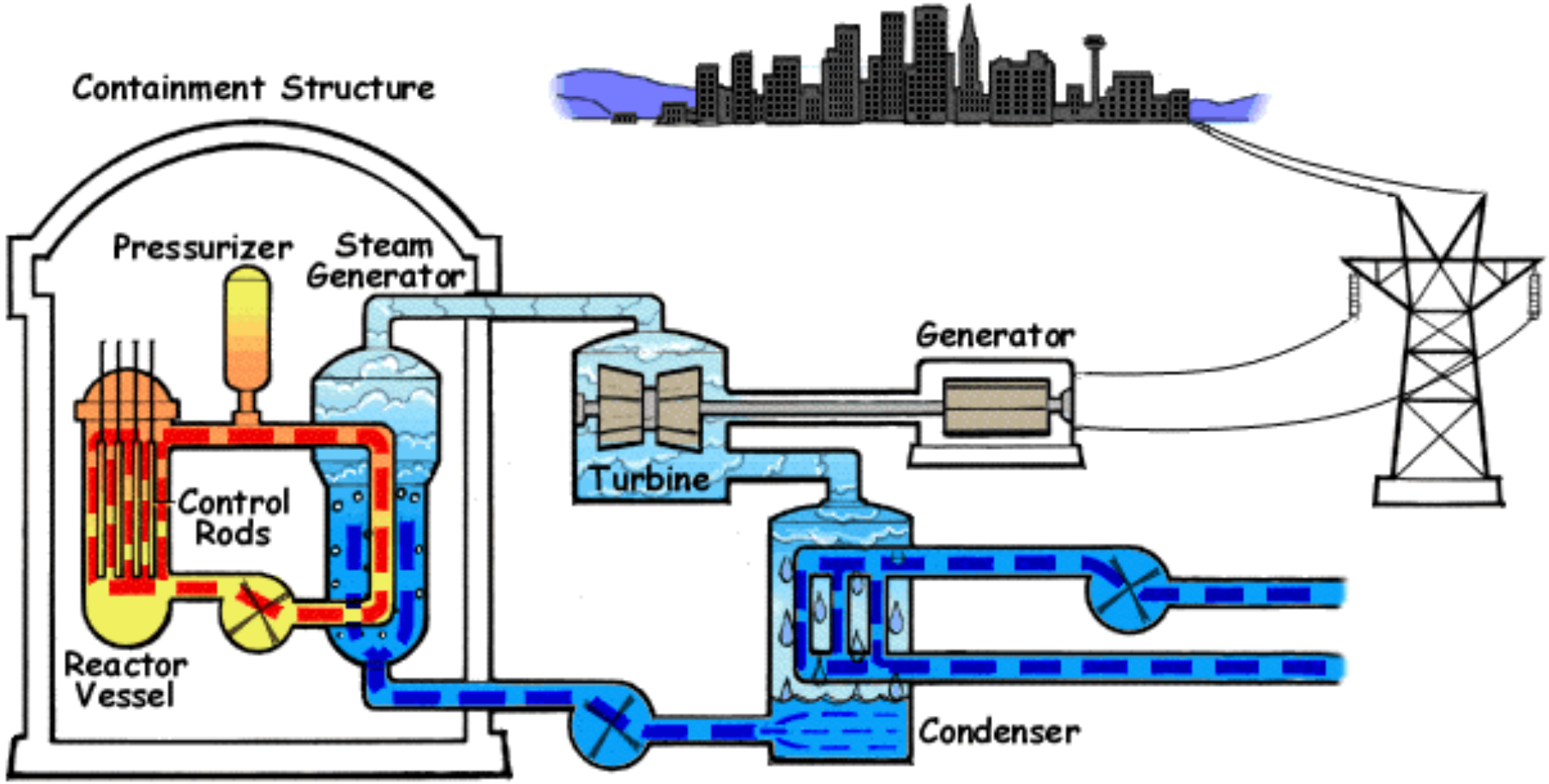
12 UTC on August 26, 2005

Cone of Uncertainty:

General Questions:

- What is expected rainfall in Research Triangle on February 25?
- What are average high and low temperatures?
- What is predicted average snow fall?
- Note: Quantities are statistical in nature.

Example 2: Pressurized Water Reactors (PWR)



Models:

- Involve neutron transport, thermal-hydraulics, chemistry.
- Inherently multi-scale, multi-physics.

CRUD Measurements: Consist of low resolution images at limited number of locations.

Example: Pressurized Water Reactors (PWR)

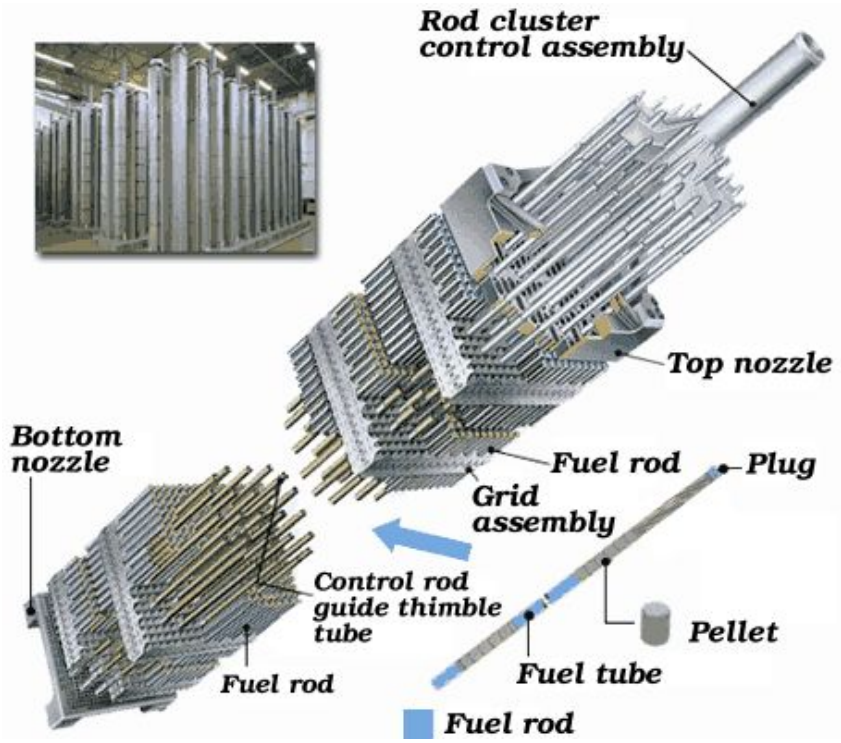
Challenges:

• Models linear in the state but function of 7 independent variables:

$$r = x, y, z; E; \Omega = \theta, \phi; t$$

• Very large number of inputs or parameters; e.g., 100,000. Parameter selection critical.

• Codes can take hours to days to run.



Example: Shearon Harris outside Raleigh

UQ Questions:

• What is peak operating temperature?

• What is expected level of CRUD buildup?

• What is associated risk?

• What is expected profit for new design?

Example 3: HIV Model for Characterization and Control Regimes

HIV Model:

$$\dot{T}_1 = \lambda_1 - d_1 T_1 - (1-\varepsilon) k_1 V T_1$$

$$\dot{T}_2 = \lambda_2 - d_2 T_2 - (1-f\varepsilon) k_2 V T_2$$

$$\dot{T}_1^* = (1-\varepsilon) k_1 V T_1 - \delta T_1^* - m_1 E T_1^*$$

$$\dot{T}_2^* = (1-f\varepsilon) k_2 V T_2 - \delta T_2^* - m_2 E T_2^*$$

$$\dot{V} = N_T \delta (T_1^* + T_2^*) - cV - [(1-\varepsilon)\rho_1 k_1 T_1 + (1-f\varepsilon)\rho_2 k_2 T_2] V$$

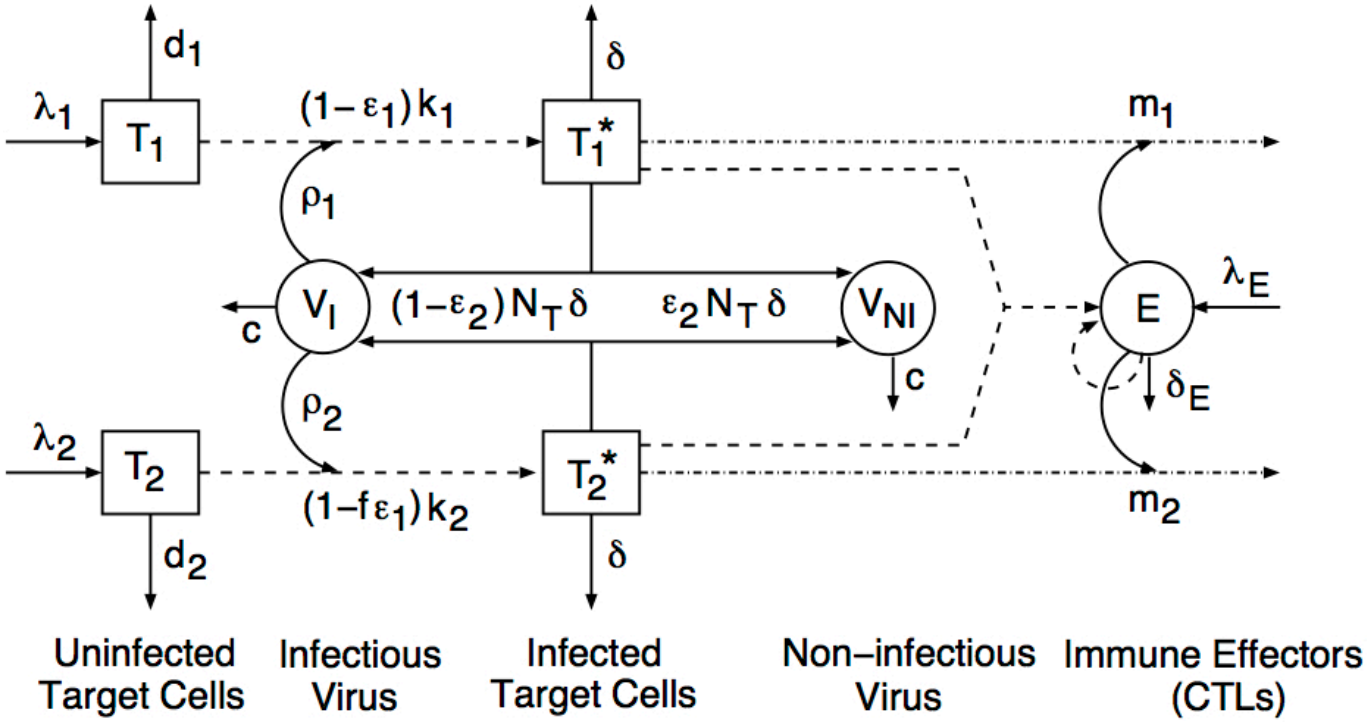
$$\dot{E} = \lambda_E + \frac{b_E (T_1^* + T_2^*)}{T_1^* + T_2^* + K_b} E - \frac{d_E (T_1^* + T_2^*)}{T_1^* + T_2^* + K_d} E - \delta_E E$$

Notes: 21 parameters

[Adams, Banks et al., 2005, 2007]

Notation: $\dot{E} \equiv \frac{dE}{dt}$

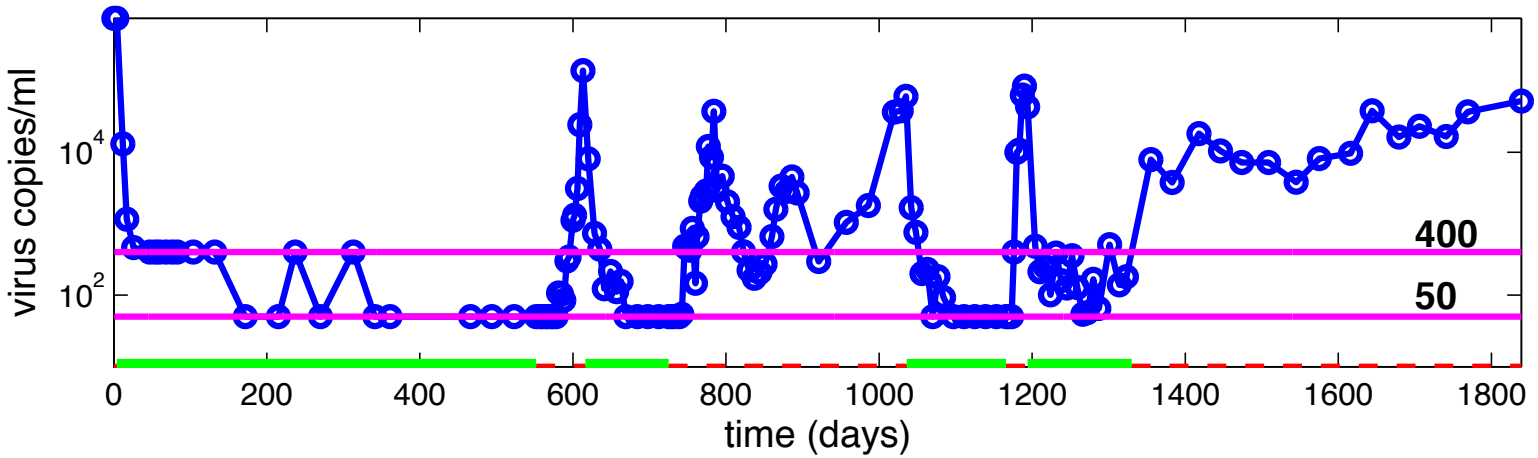
Compartments:



Example: HIV Model for Characterization and Treatment Regimes

HIV Model: Several sources of uncertainty including viral measurement techniques

Example: Upper and lower limits to assay sensitivity



UQ Questions:

- What are the uncertainties in parameters that cannot be directly measured?
- What is optimal treatment regime that is "safe" for patient?
- What is expected viral load? Issue: very often requires high-dimensional integration!
- e.g., $\mathbb{E}[V(t)] = \int_{\mathbb{R}^{21}} V(t,q)\rho(q)dq$

Experimental results are believed by everyone, except for the person who ran the experiment, source anonymous, quoted by Max Gunzburger, Florida State University.

Example 4: SIR Cholera Model

Model:

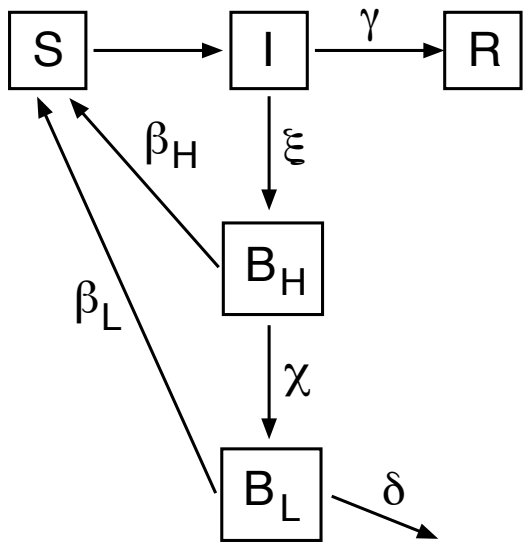
$$\frac{dS}{dt} = bN - \beta_L S\frac{B_L}{\kappa_L + B_L} - \beta_H S\frac{B_H}{\kappa_H + B_H} - bS$$

$$\frac{dI}{dt} = \beta_L S\frac{B_L}{\kappa_L + B_L} + \beta_H S\frac{B_H}{\kappa_H + B_H} - (\gamma + b)I$$

$$\frac{dR}{dt} = \gamma I - bR$$

$$\frac{dB_H}{dt} = \xi I - \chi B_H$$

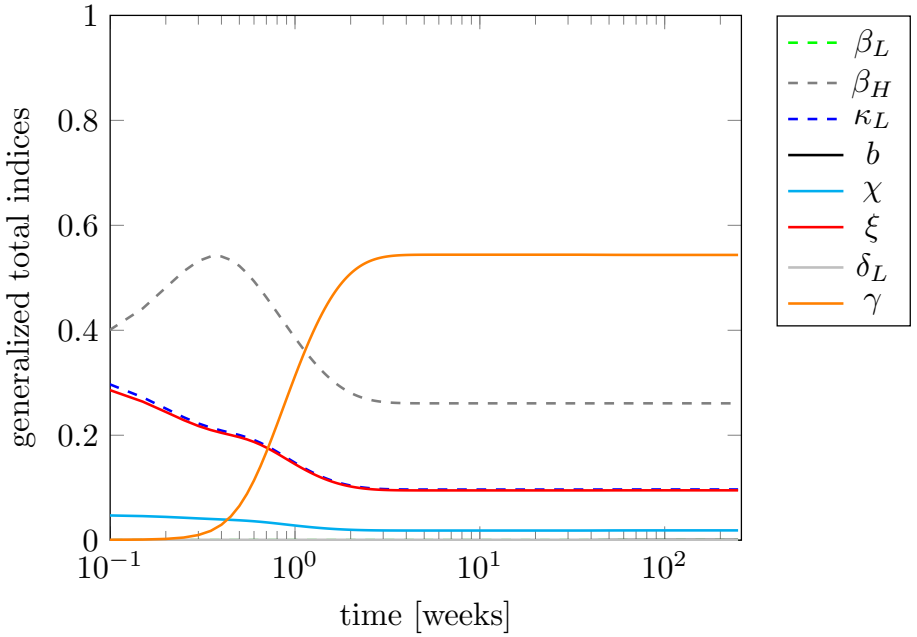
$$\frac{dB_L}{dt} = \chi B_H - \delta B_L$$



Model Parameter	Symbol	Units	Values
Rate of drinking B_L cholera	β_L	$\frac{1}{\text{week}}$	1.5
Rate of drinking B_H cholera	β_H	$\frac{1}{\text{week}}$	7.5 $(*)$
B_L cholera carrying capacity	κ_L	$\frac{\#\text{ bacteria}}{m\ell}$	10^6
B_H cholera carrying capacity	κ_H	$\frac{\#\text{ bacteria}}{m\ell}$	$\frac{\kappa_L}{700}$
Human birth and death rate	b	$\frac{1}{\text{week}}$	$\frac{1}{1560}$
Rate of decay from B_H to B_L	χ	$\frac{1}{\text{week}}$	$\frac{168}{5}$
Rate at which infectious individuals spread B_H bacteria to water	ξ	$\frac{\#\text{ bacteria}}{\#\text{ individuals}\cdot m\ell\cdot\text{week}}$	70
Death rate of B_L cholera	δ	$\frac{1}{\text{week}}$	$\frac{7}{30}$
Rate of recovery from cholera	γ	$\frac{1}{\text{week}}$	$\frac{7}{5}$

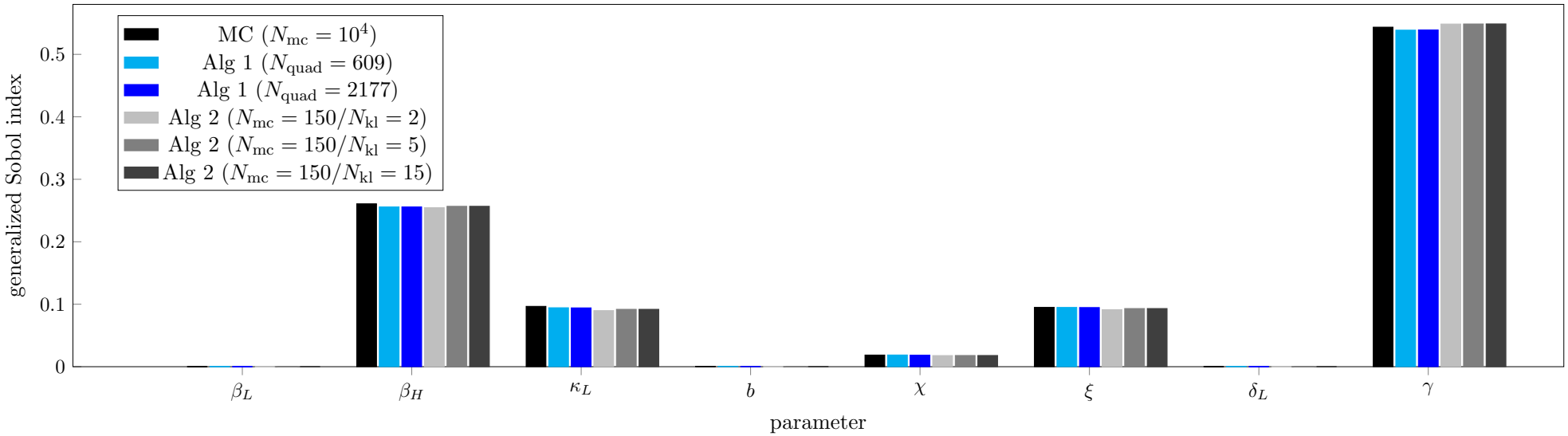
Example: SIR Cholera Model

Strategy: Time-dependent global sensitivity indices; later talk by Pierre Gremaud



Conclusion: Sensitive indices

- γ: Recovery rate
- β_H: Rate of drinking B_H cholera
- κ_L: B_L carrying capacity; Note $\kappa_H = \kappa_L/700$
- ξ: Rate at which B_H bacteria spread



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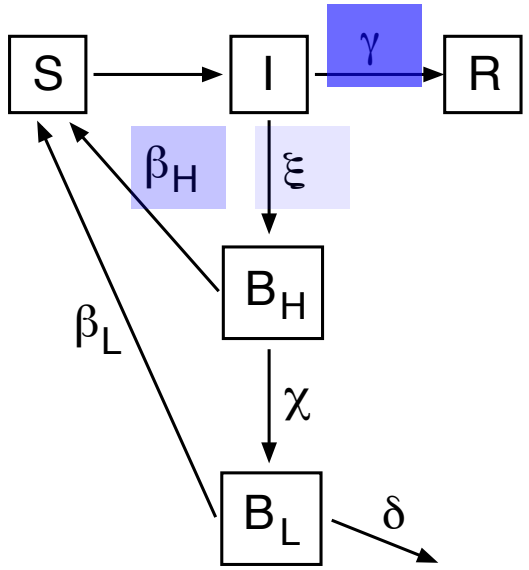
$$\frac{dB_H}{dt} = \xi I - \chi B_H$$

$$\frac{dB_L}{dt} = \chi B_H - \delta B_L$$



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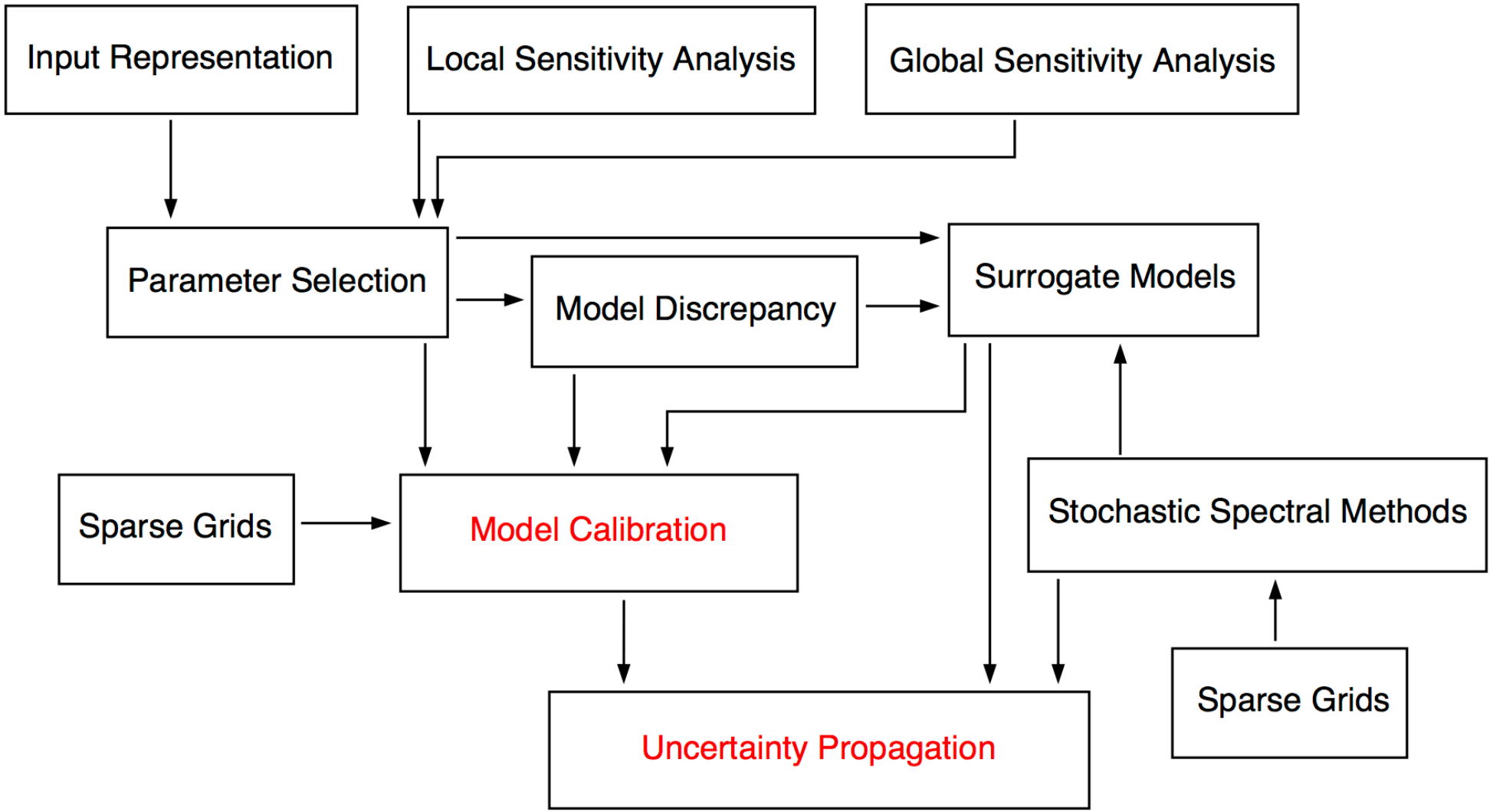
Steps in Uncertainty Quantification

Note: Uncertainty quantification requires synergy between statistics, mathematics and application area.



Model Calibration and Uncertainty Propagation

Sources of Uncertainty:

- Model
- Parameters
- Sensor measurements
- Initial conditions

Strategy:

- Quantify uncertainty in parameters
- Propagate uncertainty through model

Example: HIV model

$$\dot{T}_1 = \lambda_1 - d_1 T_1 - (1-\varepsilon) k_1 V T_1$$

$$\dot{T}_2 = \lambda_2 - d_2 T_2 - (1-f\varepsilon) k_2 V T_2$$

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$$\dot{V} = N_T \delta (T_1^* + T_2^*) - cV - [(1-\varepsilon)\rho_1 k_1 T_1 + (1-f\varepsilon)\rho_2 k_2 T_2] V$$

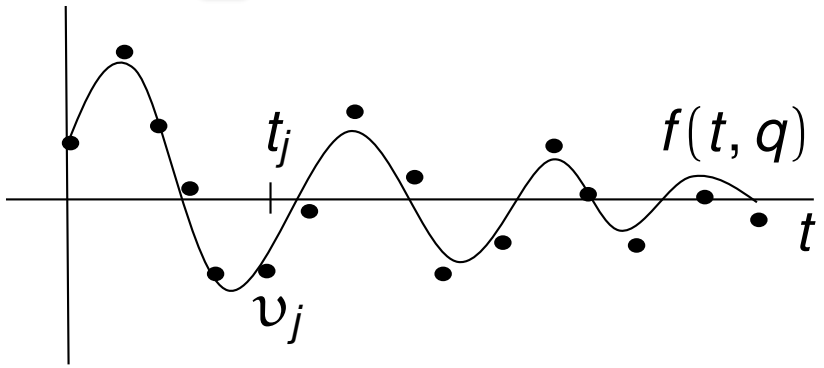
$$\dot{E} = \lambda_E + \frac{b_E (T_1^* + T_2^*)}{T_1^* + T_2^* + K_b} E - \frac{d_E (T_1^* + T_2^*)}{T_1^* + T_2^* + K_d} E - \delta_E E$$

Parameters: Reduced set

$$q = [b_E, \delta, d_1, k_2, \lambda_1, K_b]$$

Point Estimates: Ordinary least squares

$$q^0 = \arg\min_q \frac{1}{2} \sum_{j=1}^{N} [\upsilon_j - f(t_j, q)]^2$$



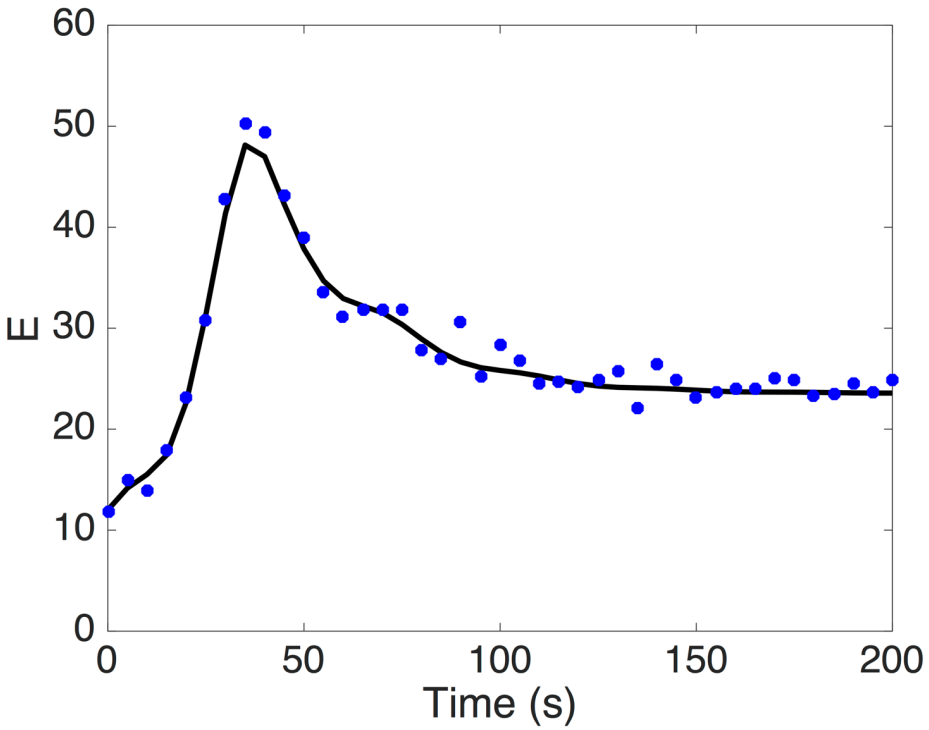
Note: Scaling critical since parameter values vary by 8 orders of magnitude.

Model Calibration and Predictions

Optimization Results:

b_E	δ	d_1	k_2	λ_1	K_b
0.30	0.68	9.1×10^{-3}	1.22×10^{-4}	9.95×10^{3}	88.5

Data and Prediction of Immune Effector Response E:



Note: Point estimates but no quantification of uncertainty in:

- Model
- Parameters
- Data

Goals:

- Replace point estimates with distributions.
- Construct credible and prediction intervals.
- Natural in a Bayesian framework

Statistical Inference

Goal: The goal in statistical inference is to make conclusions about a phenomenon based on observed data.

Frequentist: Observations made in the past are analyzed with a specified model. Result is regarded as confidence about state of real world.

- Probabilities defined as frequencies with which an event occurs if experiment is repeated several times.
- Parameter Estimation:
    - Relies on estimators derived from different data sets and a specific sampling distribution.
    - Parameters may be unknown but are fixed and deterministic.

Bayesian: Interpretation of probability is subjective and can be updated with new data.

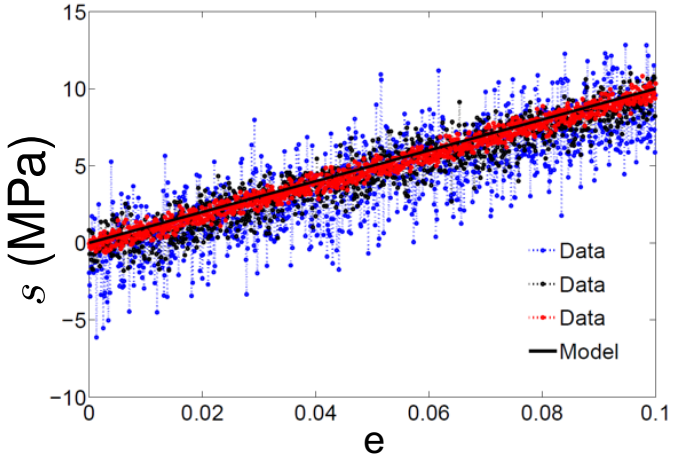
- Parameter Estimation: Parameters are considered to be random variables having associated densities.

Bayesian Inference: More General Model

Example: Displacement-force relation (Hooke's Law)

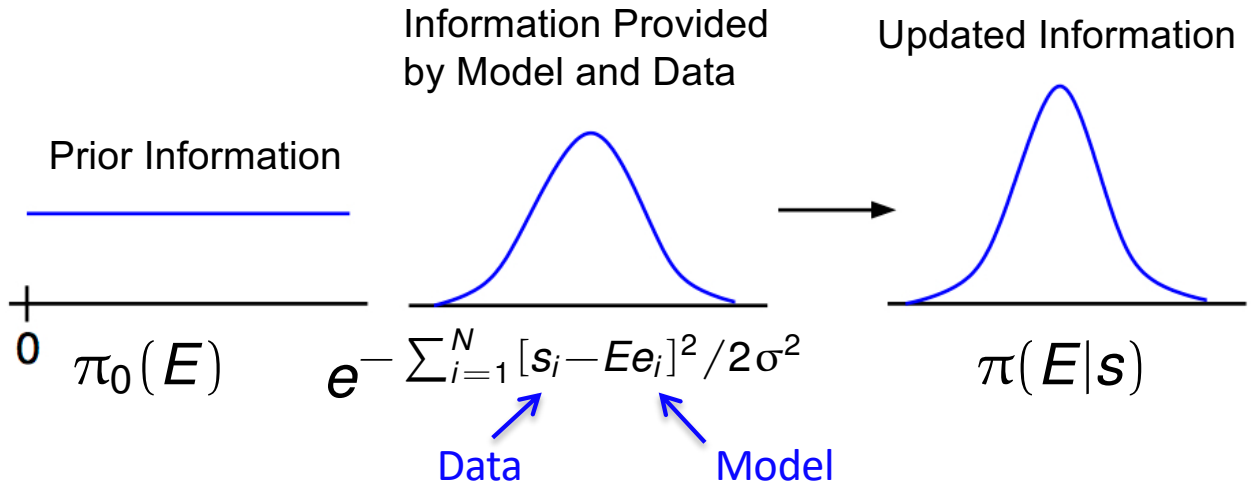
$$s_i = Ee_i + \varepsilon_i \ , \ i = 1, \ldots, N$$

$$\varepsilon_i \sim N(0, \sigma^2)$$



Parameter: Stiffness E

Strategy: Use model fit to data to update prior information

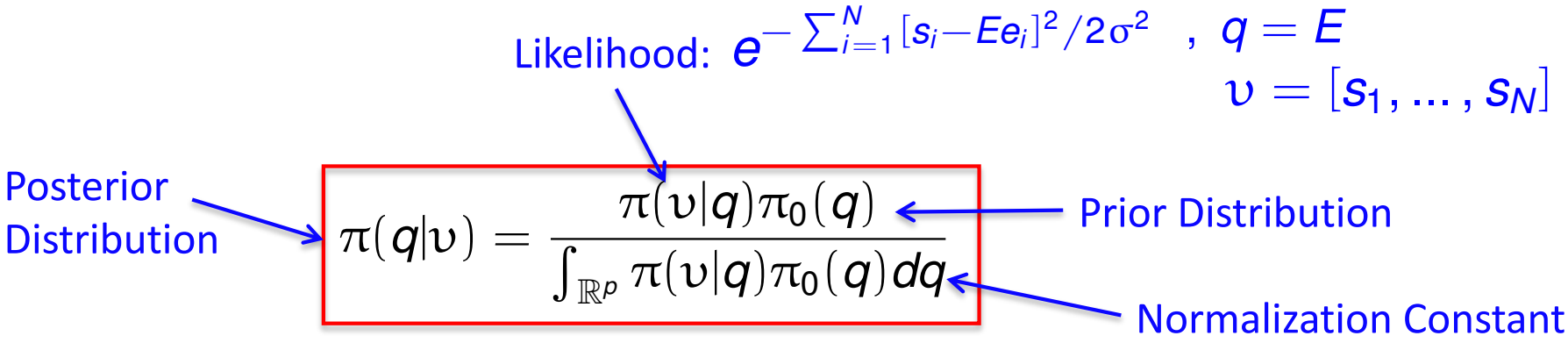


Non-normalized Bayes' Relation:

$$\pi(E|s) = e^{-\sum_{i=1}^{N}[s_i - Ee_i]^2/2\sigma^2}\pi_0(E)$$

Bayesian Inference

Bayes' Relation: Specifies posterior in terms of likelihood and prior



- Prior Distribution: Quantifies prior knowledge of parameter values
- Likelihood: Probability of observing a data given set of parameter values.
- Posterior Distribution: Conditional distribution of parameters given observed data.

Problem: Can require high-dimensional integration

- e.g., HIV Model: p = 6 - 23!
- Solution: Sampling-based Markov Chain Monte Carlo (MCMC) algorithms.
- Metropolis algorithms first used by nuclear physicists during Manhattan Project in 1940's to understand particle movement underlying first atomic bomb.

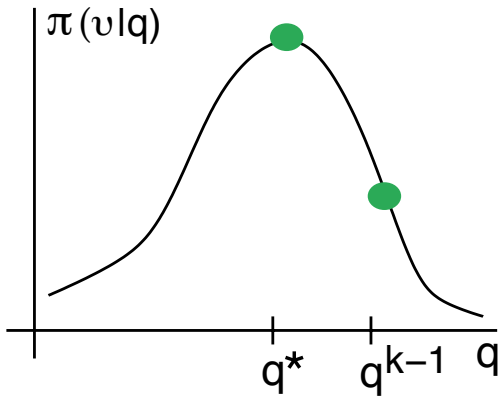
Markov Chain Monte Carlo Methods

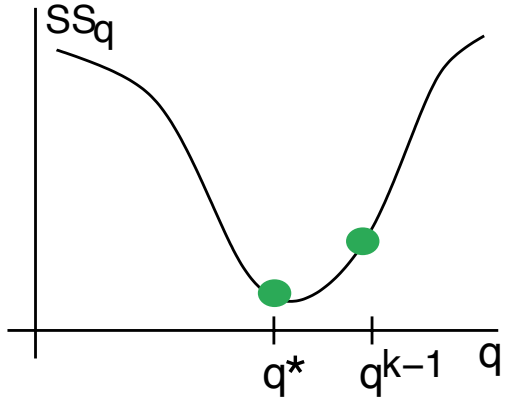
Strategy:

- Sample values from proposal distribution $J(q^*|q^{k-1})$ that reflects geometry of posterior distribution
- Compute $r(q^*|q^{k-1}) = \frac{\pi(\upsilon|q^*)\pi_0(q^*)}{\pi(\upsilon|q^{k-1})\pi_0(q^{k-1})}$
  - If $r \geqslant 1$, accept with probability $\alpha = 1$
  - If $r < 1$, accept with probability $\alpha = r$

Intuition: Consider flat prior $\pi_0(q) = 1$ and Gaussian observation model

$$\pi(\upsilon|q) = \frac{1}{(2\pi\sigma^2)^{n/2}} e^{-SS_q/2\sigma^2} \qquad SS_q = \sum_{i=1}^{N} [\upsilon_i - f(t_i, q)]^2$$

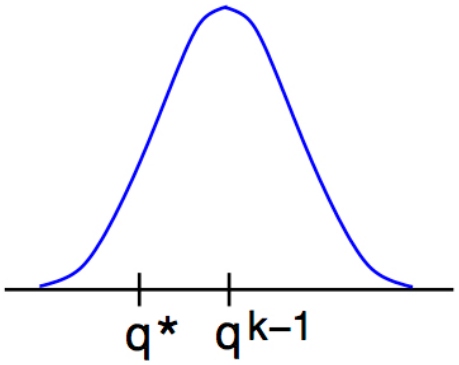




Delayed Rejection Adaptive Metropolis (DRAM)

Algorithm: [Haario et al., 2006] – MATLAB, Python

1. Determine $q^0 = \arg\min_q \sum_{i=1}^{N} [\upsilon_i - f(t_i, q)]^2$
2. For $k = 1, \cdots, M$
   (a) Construct candidate $q^* \sim N(q^{k-1}, V)$



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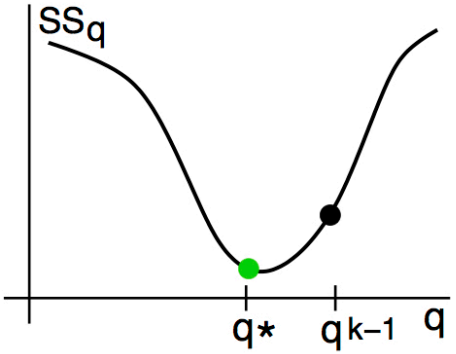

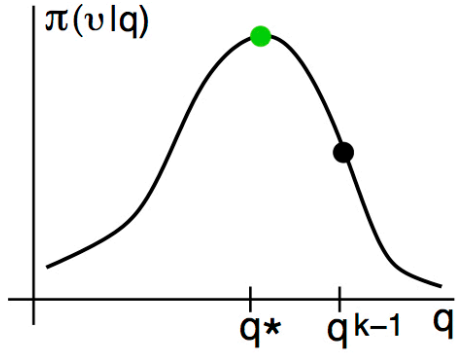

   (c) Accept q^* with probability dictated by likelihood

Delayed Rejection Adaptive Metropolis (DRAM)

Algorithm: [Haario et al., 2006] – MATLAB, Python

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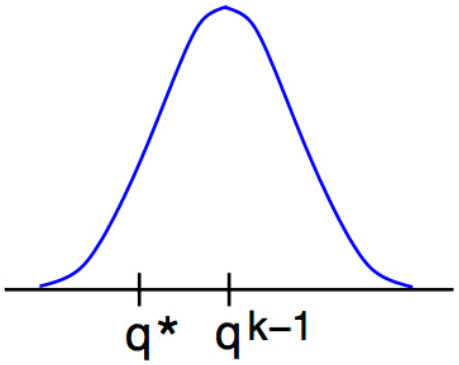

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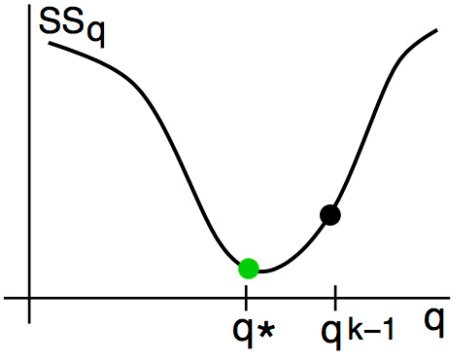

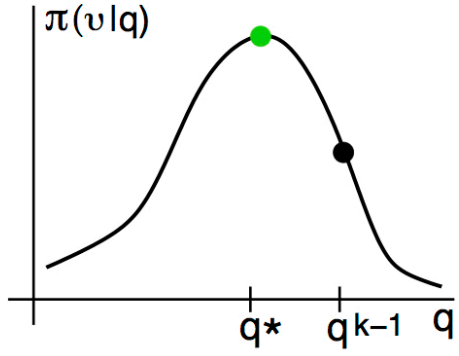

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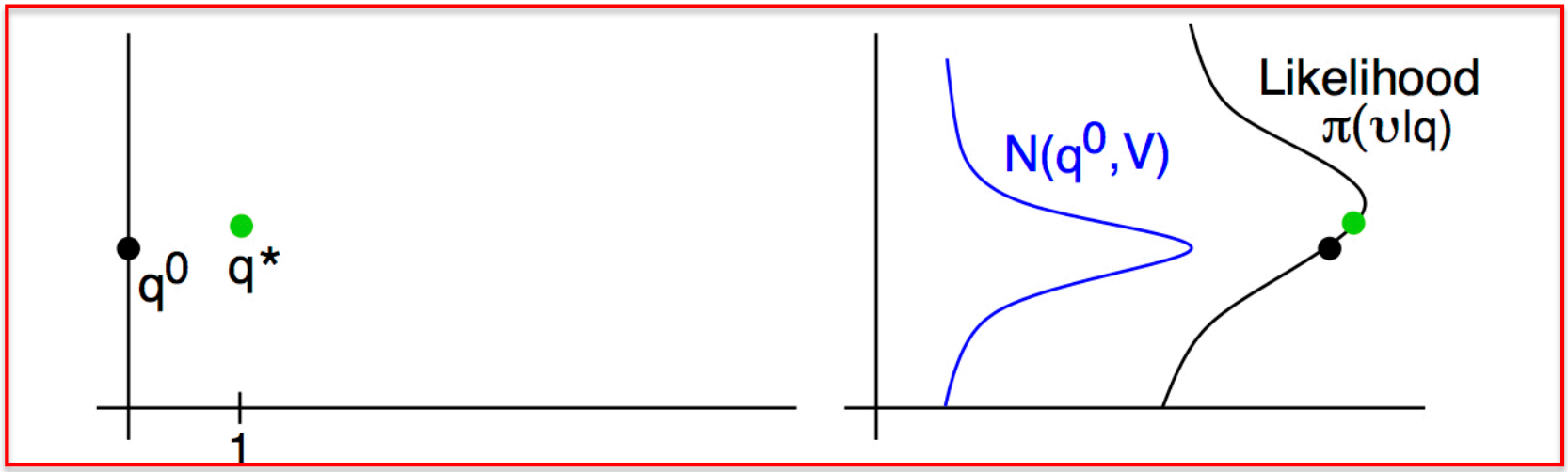

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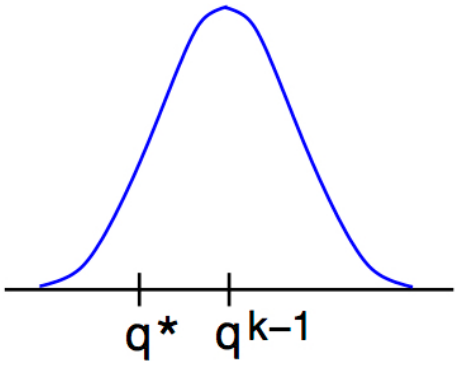

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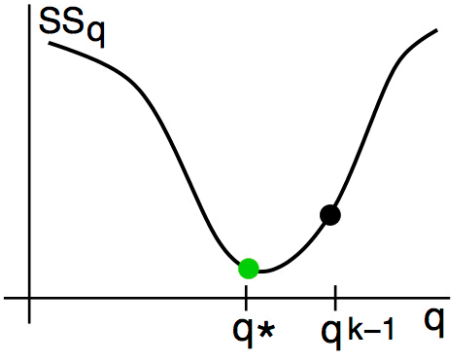

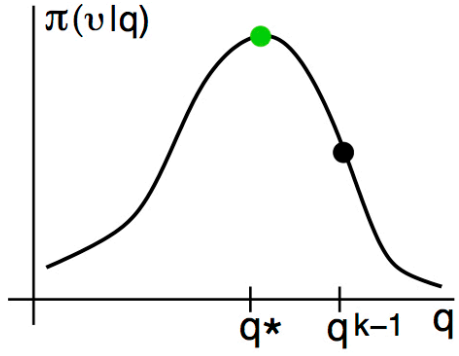

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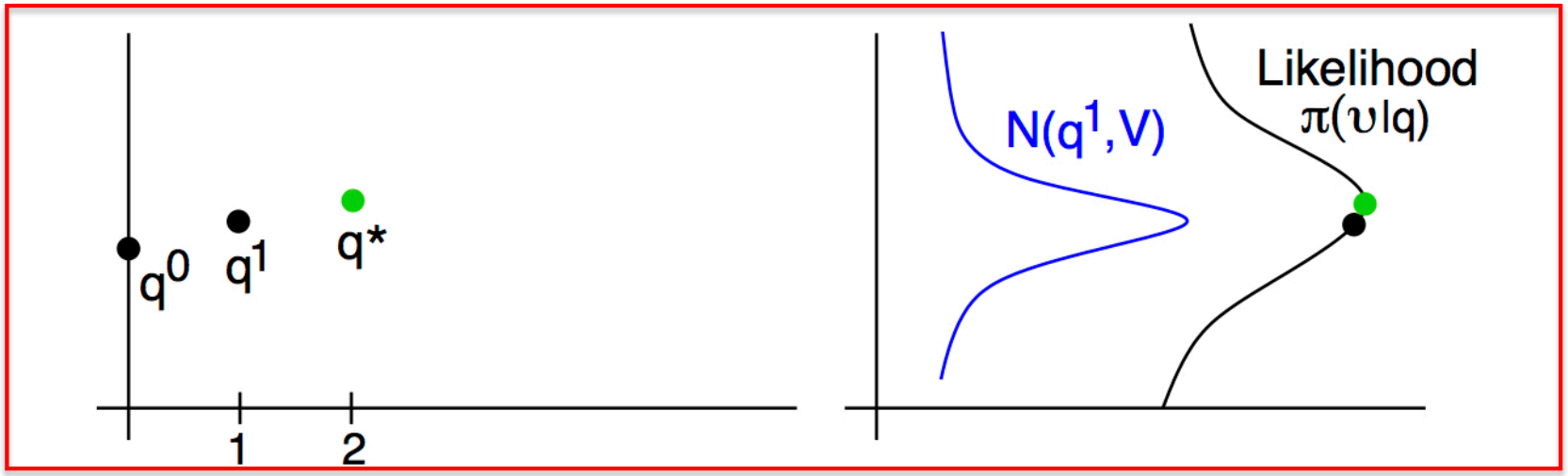

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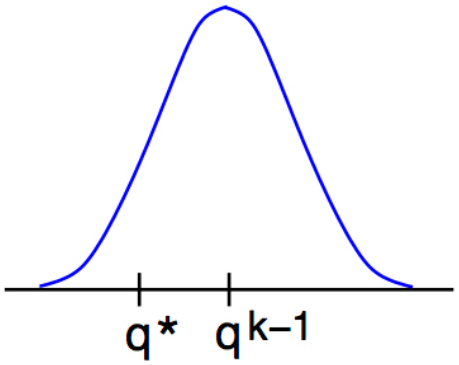

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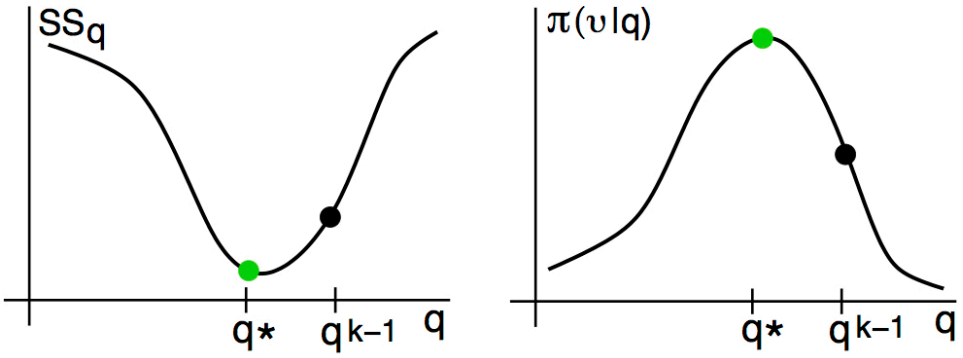

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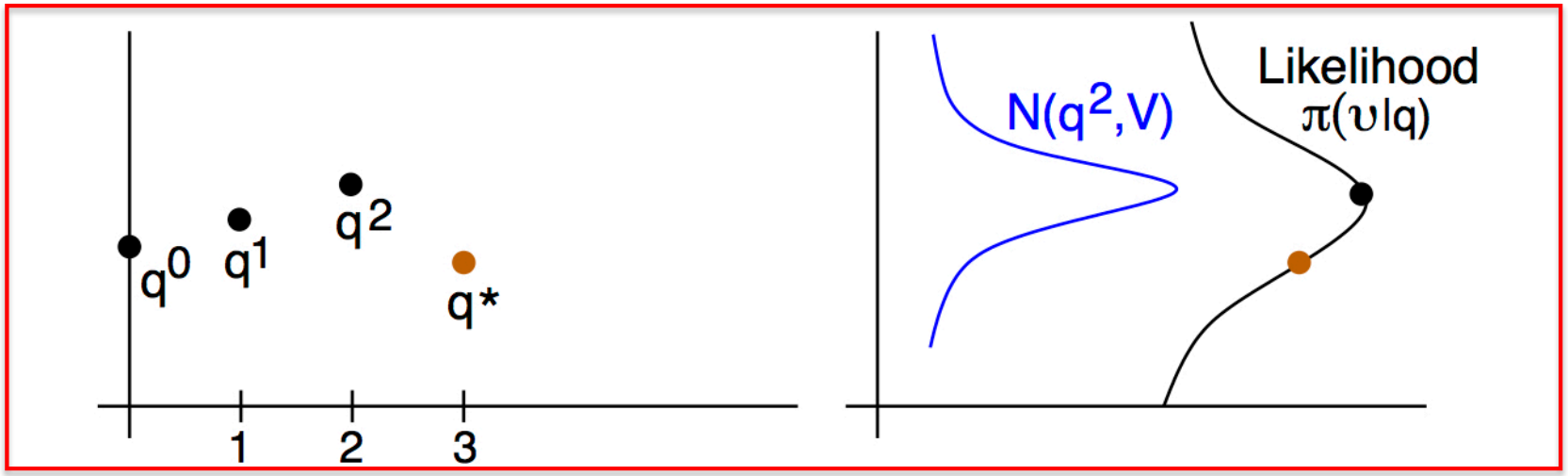

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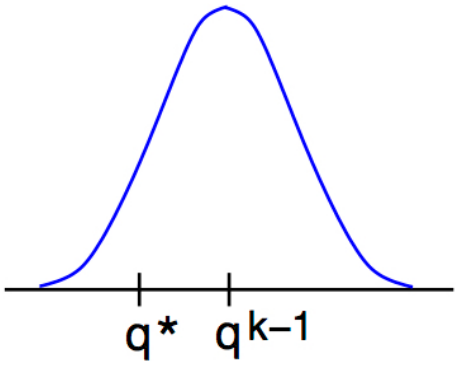

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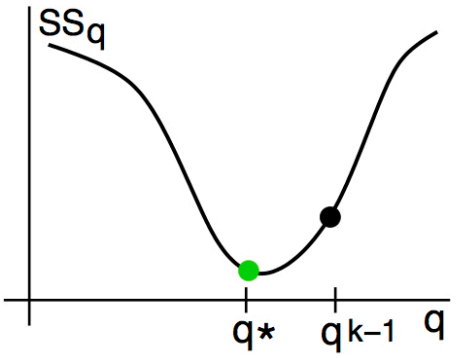

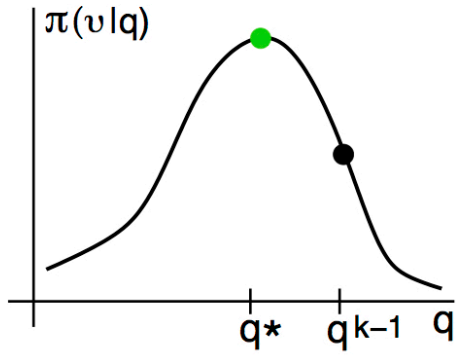

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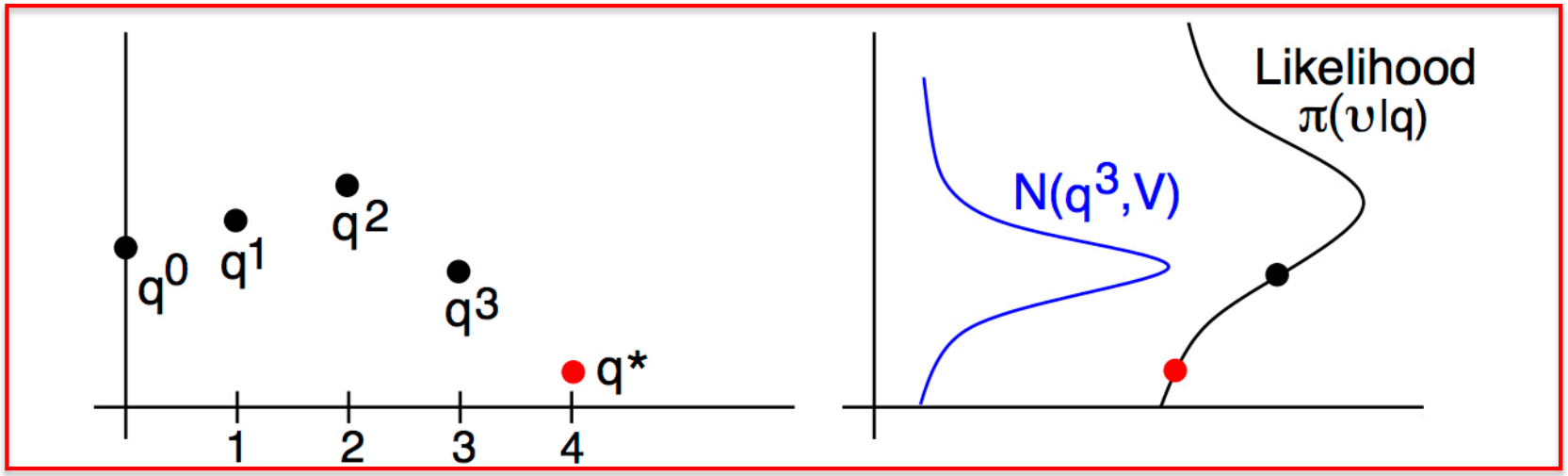

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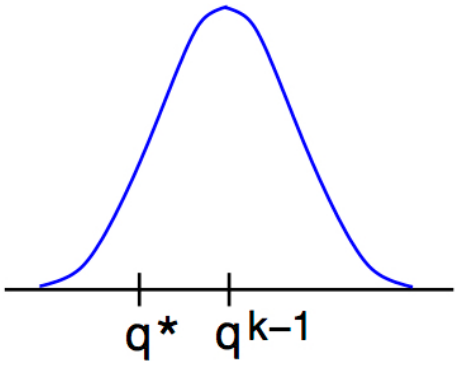

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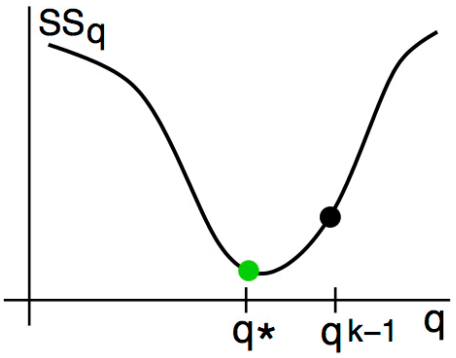

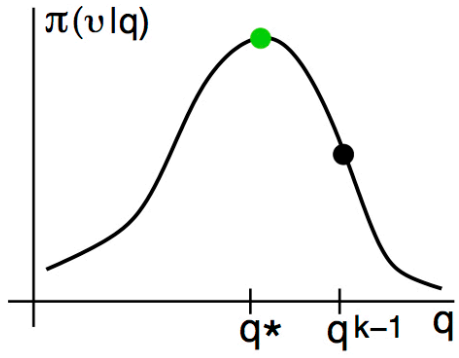

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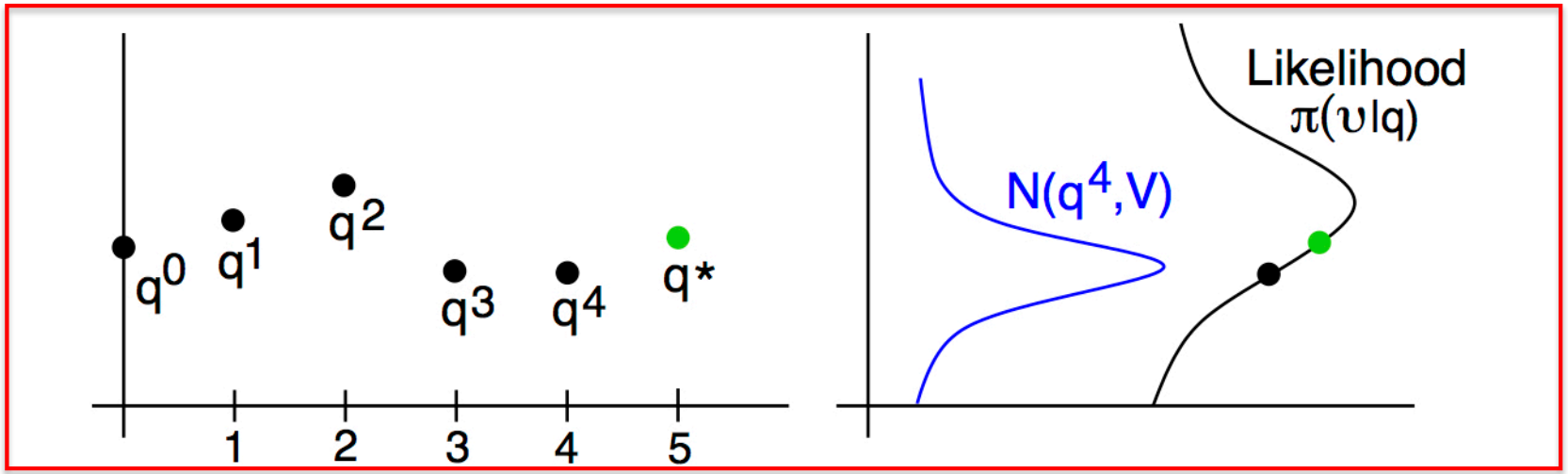


Bayesian Model Calibration – HIV Example

Model:

$$\dot{T}_1 = \lambda_1 - d_1 T_1 - (1-\varepsilon) k_1 V T_1$$

$$\dot{T}_2 = \lambda_2 - d_2 T_2 - (1-f\varepsilon) k_2 V T_2$$

$$\dot{T}_1^* = (1-\varepsilon) k_1 V T_1 - \delta T_1^* - m_1 E T_1^*$$

$$\dot{T}_2^* = (1-f\varepsilon) k_2 V T_2 - \delta T_2^* - m_2 E T_2^*$$

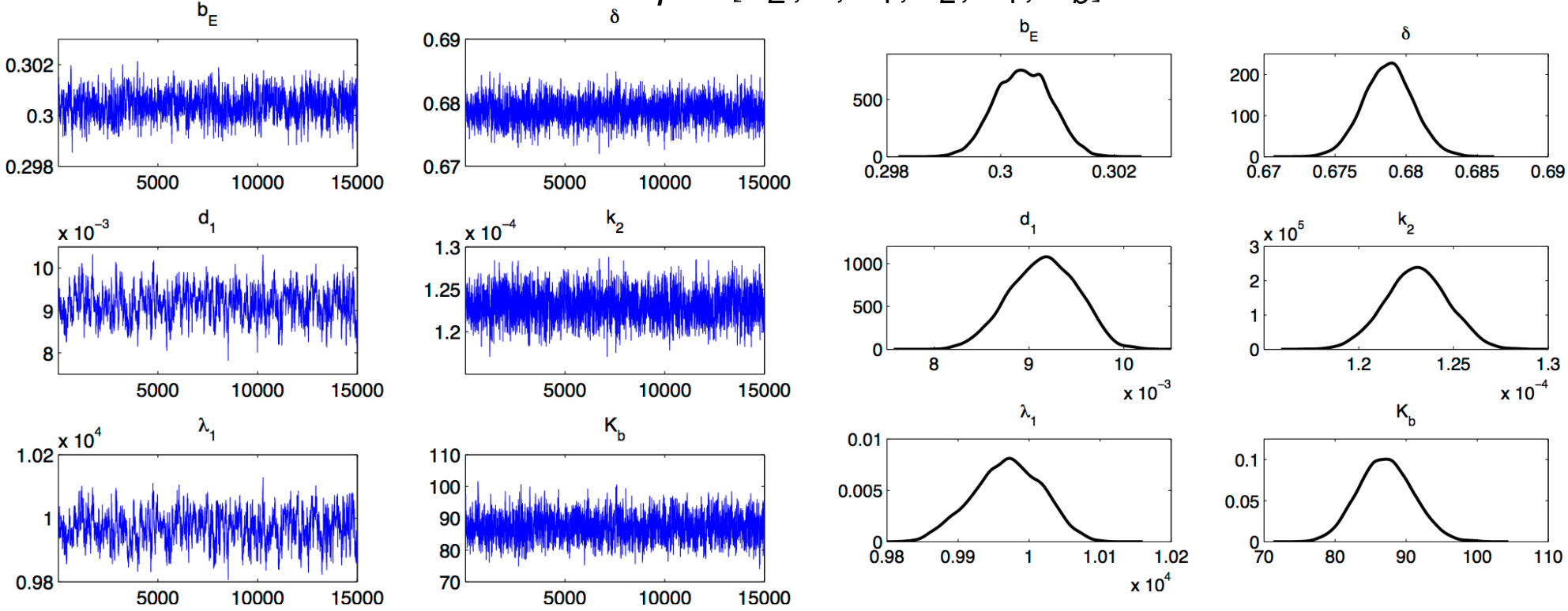
$$\dot{V} = N_T \delta (T_1^* + T_2^*) - cV - [(1-\varepsilon)\rho_1 k_1 T_1 + (1-f\varepsilon)\rho_2 k_2 T_2] V$$

$$\dot{E} = \lambda_E + \frac{b_E (T_1^* + T_2^*)}{T_1^* + T_2^* + K_b} E - \frac{d_E (T_1^* + T_2^*)}{T_1^* + T_2^* + K_d} E - \delta_E E$$

Verification: Why do we trust results??

- Compare results from different algorithms; e.g., DRAM and Gibbs

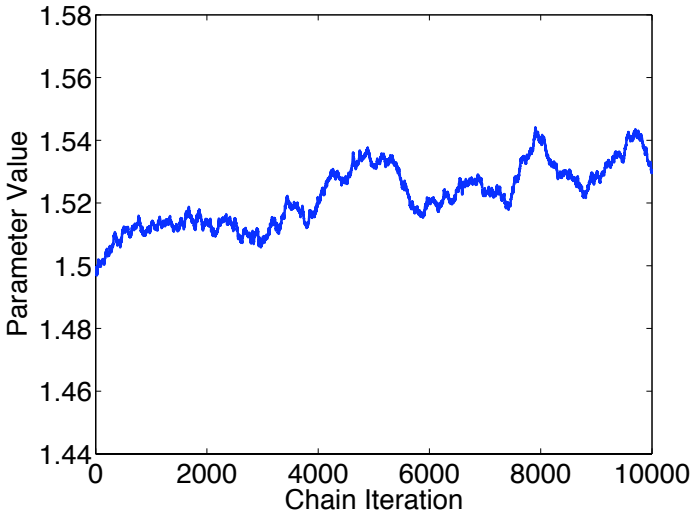
Parameter Chains and Densities: $q = [b_E, \delta, d_1, k_2, \lambda_1, K_b]$



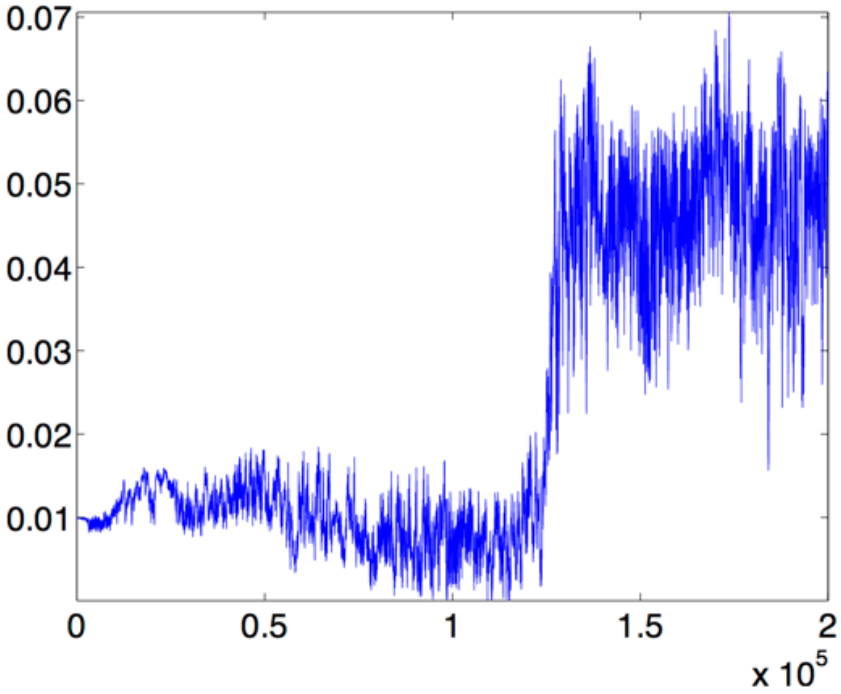
Chain Convergence (Burn-In)

Techniques:

- Visually check chains
- Statistical tests
- Often abused in the literature



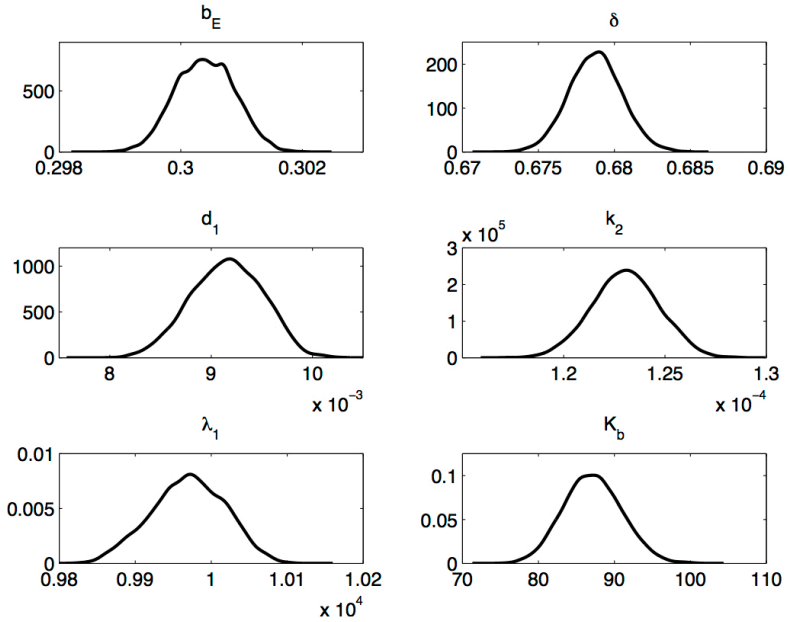
Chain not converged



Chain for nonidentifiable parameter

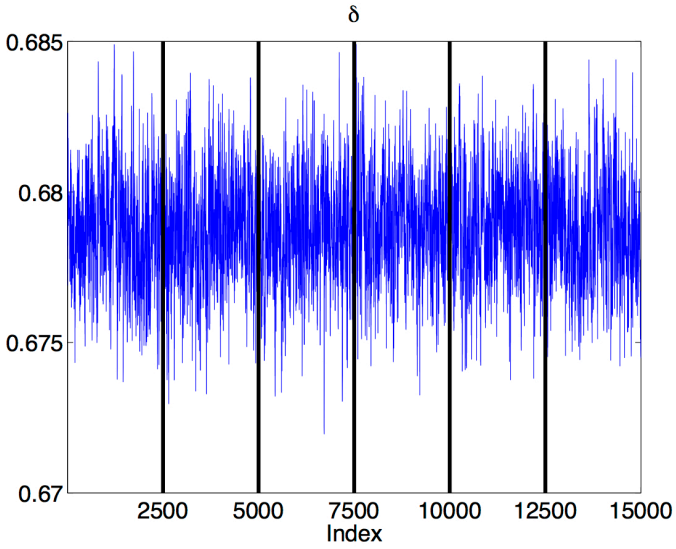
Propagation of Uncertainty in Models – HIV Example

Parameter Densities:

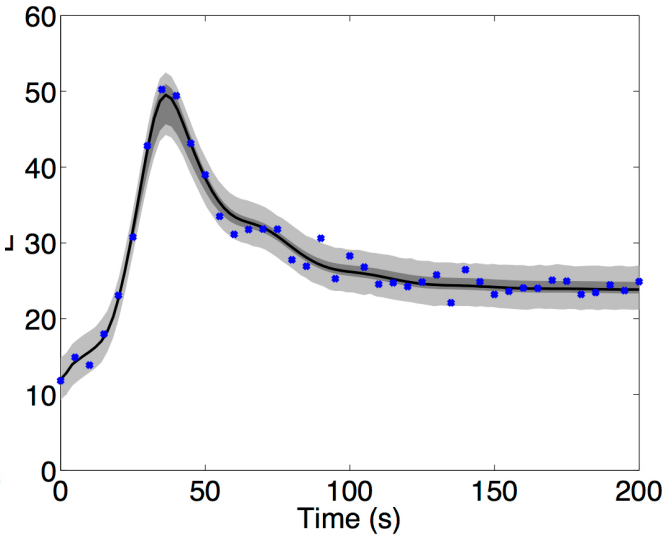


Techniques:

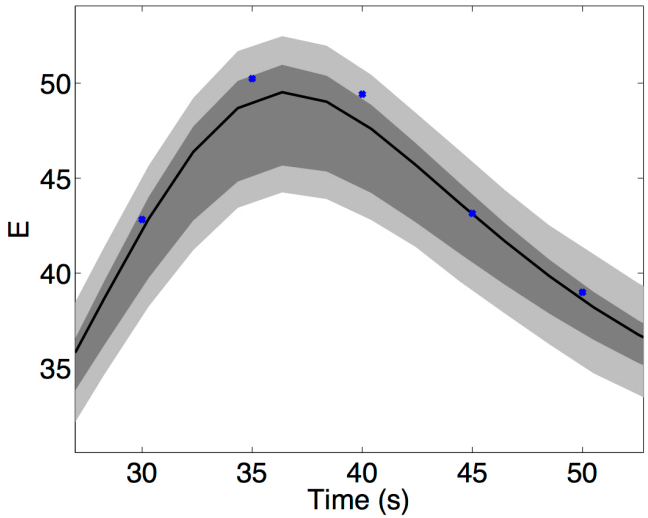
- Sample from parameter densities to construct prediction intervals for QoI.
- Slow convergence rate $O(1/\sqrt{M})$
- 100-fold more evaluations required to gain additional place of accuracy.
- Significant numerical analysis used to efficiently propagate densities.



Samples from Chain



Data, Credible Intervals and Prediction Intervals



Non-Gaussian Credible and Prediction Intervals

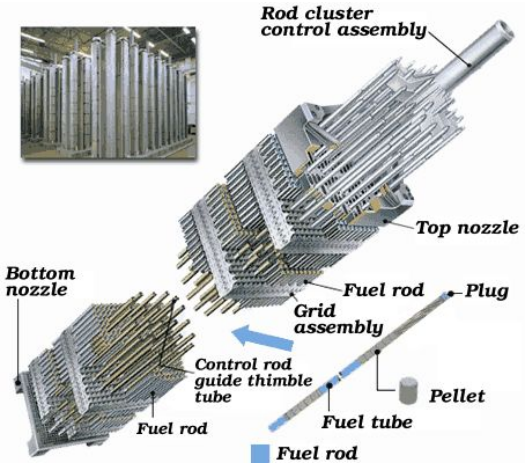
Use of Prediction Intervals: Nuclear Power Plant Design

Subchannel Code (COBRA-TF): numerous closure relations, ~70 parameters

e.g., Dittus—Boelter Relation

$$Nu = 0.023 Re^{0.8} Pr^{0.4}$$

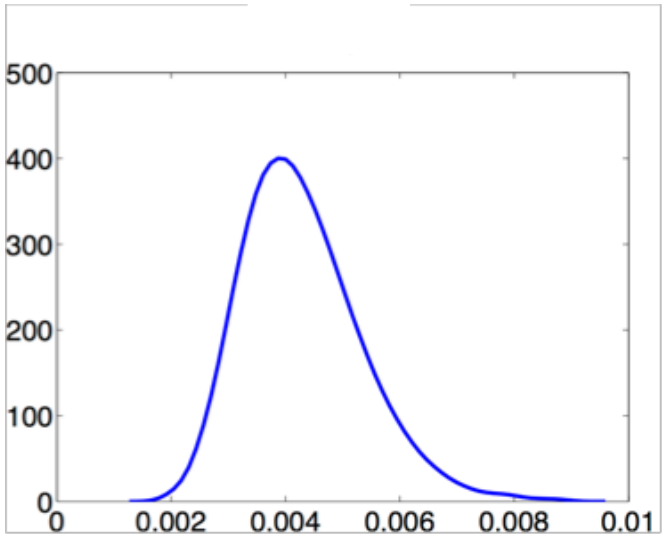
Nu: Nusselt number
Re: Reynolds number
Pr: Prandtl number



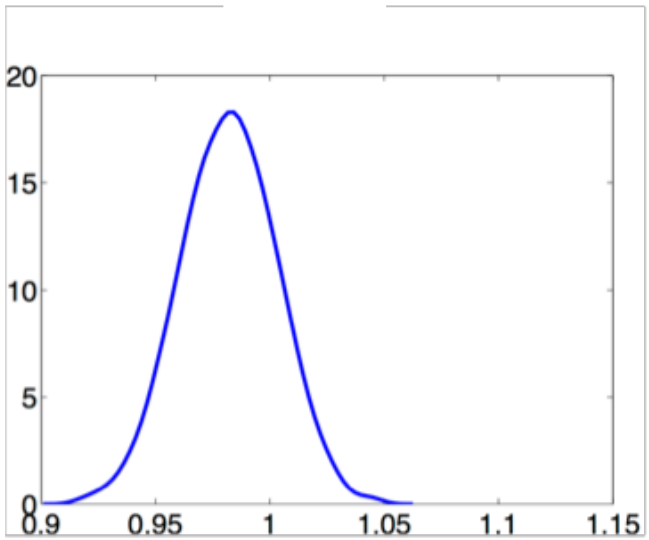
Industry Standard: Employ conservative, uniform, bounds

i.e., [0, 0.046], [0, 1.6], [0,0.8]

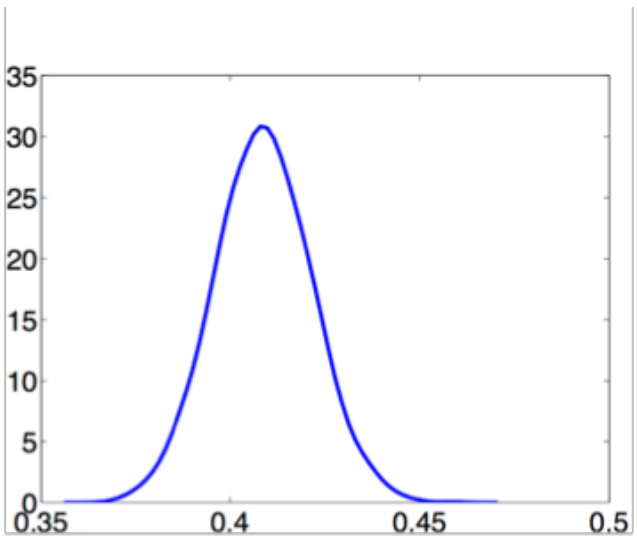
Bayesian Analysis: Employ conservative bounds as priors



$2\sigma \approx 0.0035$



$2\sigma \approx 0.06$

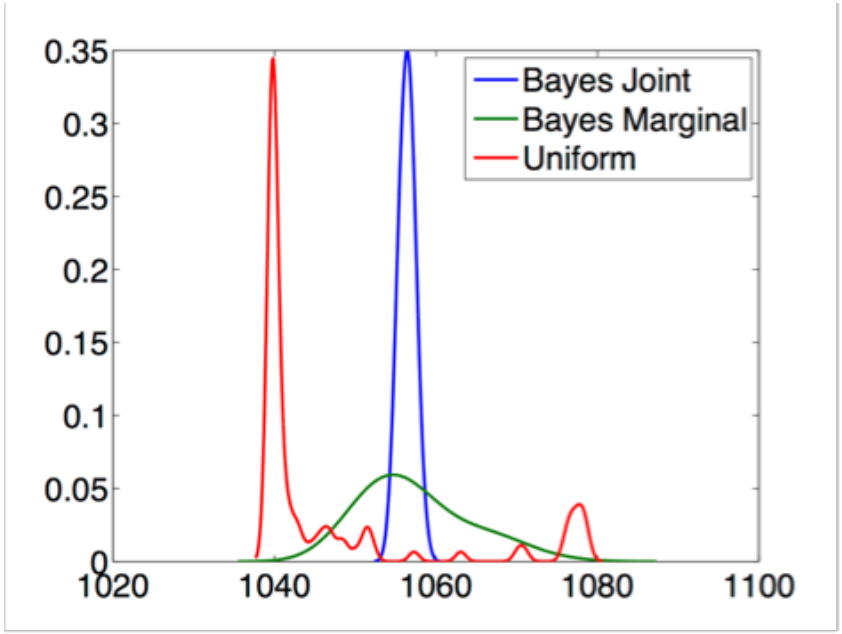


$2\sigma \approx 0.03$

Note: Substantial reduction in parameter uncertainty

Use of Prediction Intervals: Nuclear Power Plant Design

Strategy: Propagate parameter uncertainties through COBRA-TF to determine uncertainty in maximum fuel temperature



Notes:

- Temperature uncertainty reduced from 40 degrees to 5 degrees
- Can run plant 20 degrees hotter, which significantly improves efficiency

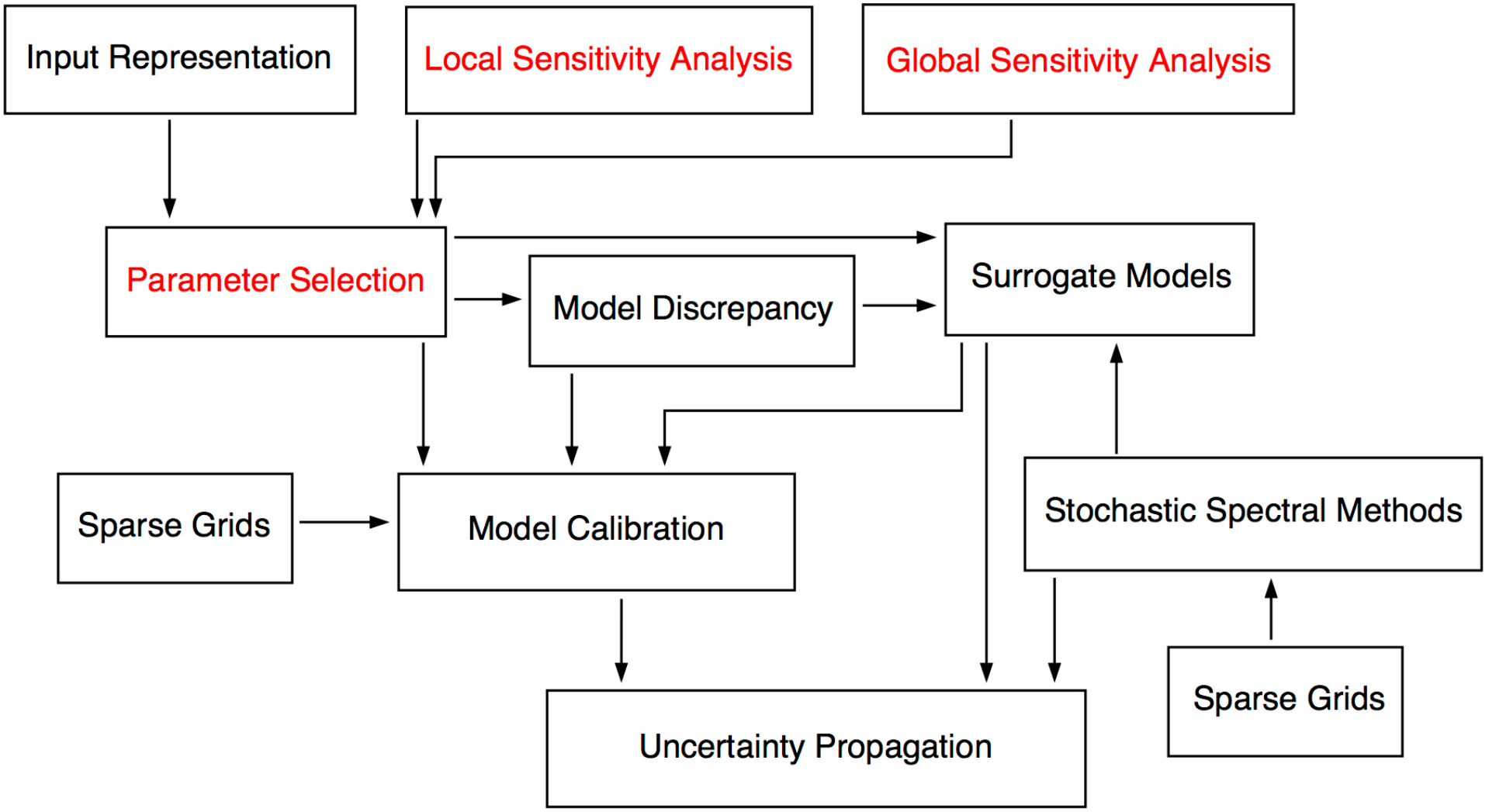
Ramification: Savings of 10 billion dollars per year for US power plants

Issues:

- We considered only one of many physical relations
- Nuclear regulatory commission takes years to change requirements and codes

Good News: We are now working with Westinghouse to reduce uncertainties.

Steps in Uncertainty Quantification



Parameter Selection: Required for models with unidentifiable or noninfluential inputs

- e.g., HIV and SIR model

Parameter Selection Techniques

Issue: Parameters often not *identifiable* in the sense that they are uniquely determined by the data.

SIR Model:

$$\frac{dS}{dt} = \delta N - \delta S - \gamma k I S \quad , \; S(0) = S_0 \qquad \text{Susceptible}$$

$$\frac{dI}{dt} = \gamma k I S - (r + \delta) I \quad , \; I(0) = I_0 \qquad \text{Infectious}$$

$$\frac{dR}{dt} = rI - \delta R \quad , \; R(0) = R_0 \qquad \text{Recovered}$$

Response:

$$y = \int_0^5 R(t, q)\, dt$$

Note: Parameter set $q = [\gamma, k, r, \delta]$ is not identifiable

Later Talk: Pierre Gremaud -- *A Biased Introduction to Global Sensitivity Analysis*

SIR Disease Example

SIR Model:

$$\frac{dS}{dt} = \delta N - \delta S - \gamma k I S \quad , \; S(0) = S_0 \qquad \text{Susceptible}$$

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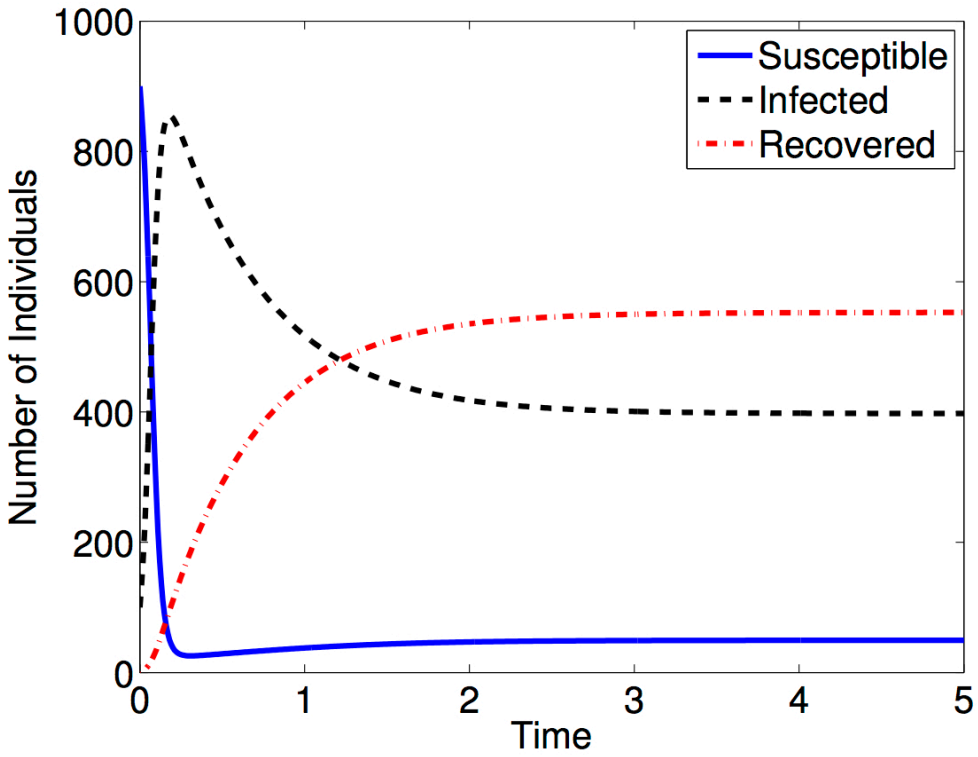
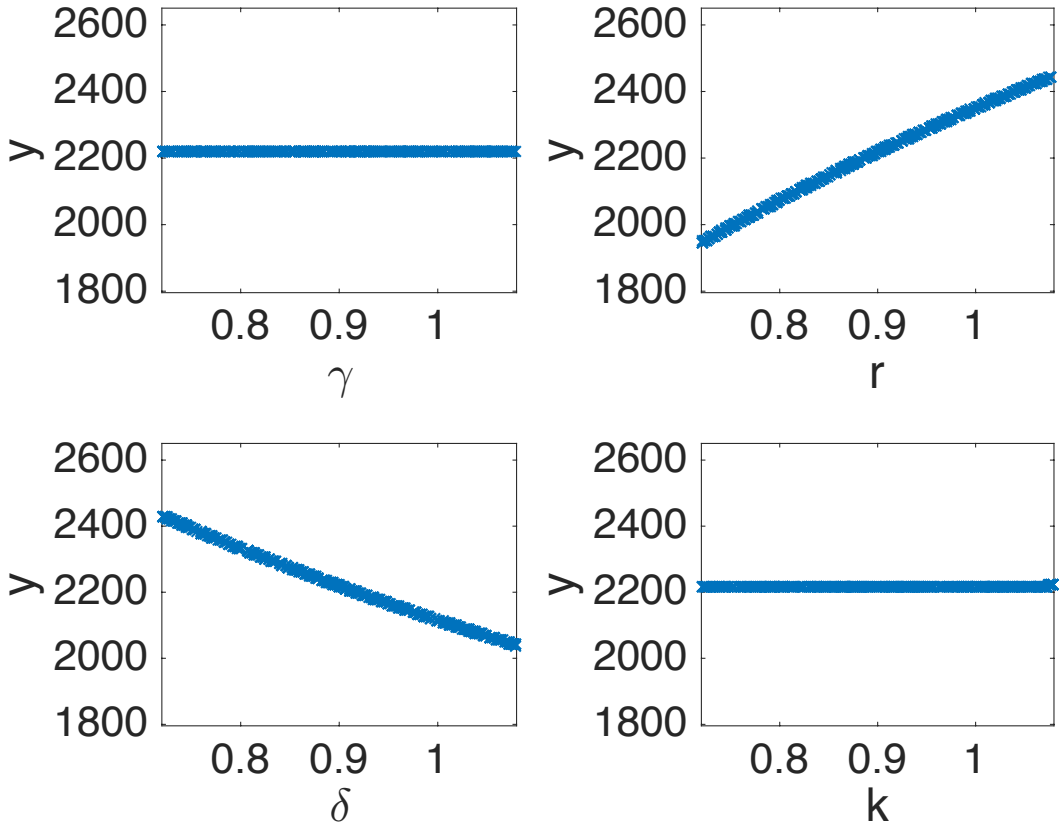
Typical Realization:



Local Sensitivity Analysis

Local Sensitivities: Consider

$$\frac{\partial y}{\partial \gamma}, \frac{\partial y}{\partial r}, \frac{\partial y}{\partial \delta}, \frac{\partial y}{\partial k}$$



Conclusion: Response most sensitive to r and δ

Limitations:

- Does not accommodate potential uncertainty in parameters.
- Sensitive to units and magnitudes of parameters.

Global Sensitivity Analysis

Global Sensitivities: Sample parameters from uniform distributions; e.g.,

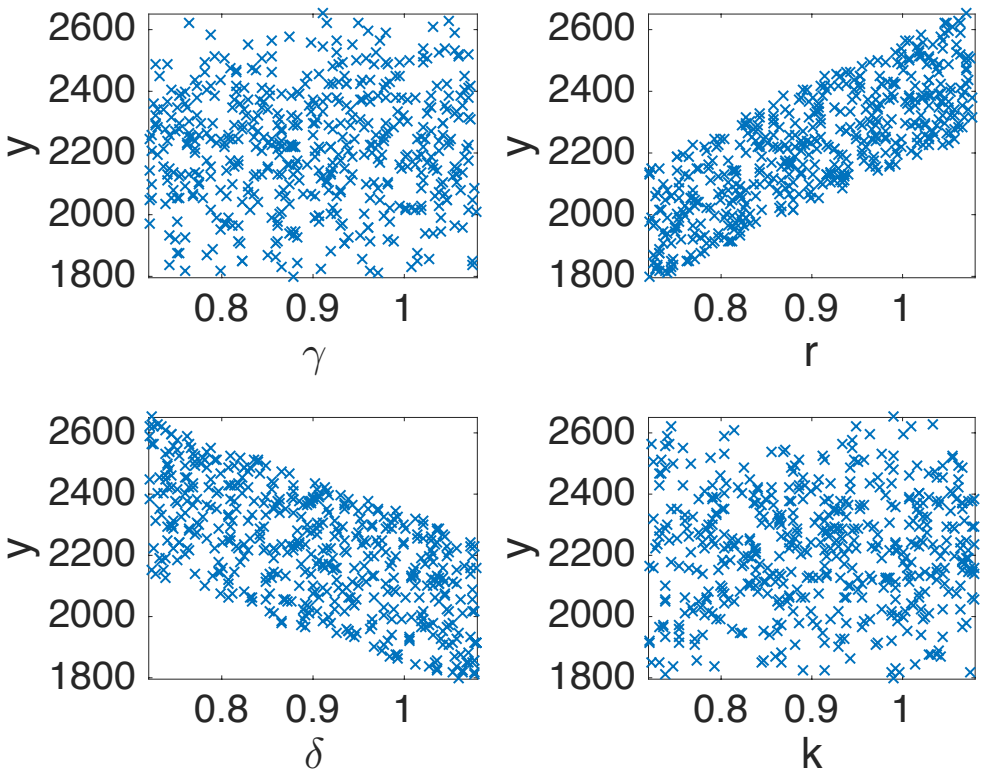
$$\gamma \sim \mathcal{U}(\gamma_\ell, \gamma_r)$$

$$\gamma_\ell = \gamma_{nom} - 0.2\gamma_{nom}$$

$$\gamma_r = \gamma_{nom} + 0.2\gamma_{nom}$$

Recall: MATLAB command to sample M samples from U(a,b)

$$>> q = a + (b - a) * \text{rand}(M, 1)$$



Conclusion: Response most sensitive to *r* and δ

Advantage:

- Quantifies how uncertainties in response apportioned to uncertainties in parameters -- Basis for Analysis of Variance (ANOVA).

Variance-Based Methods

Sobol Representation: For now, take $Q_i \sim \mathcal{U}(0,1)$ and $\Gamma = [0,1]^p$

Take

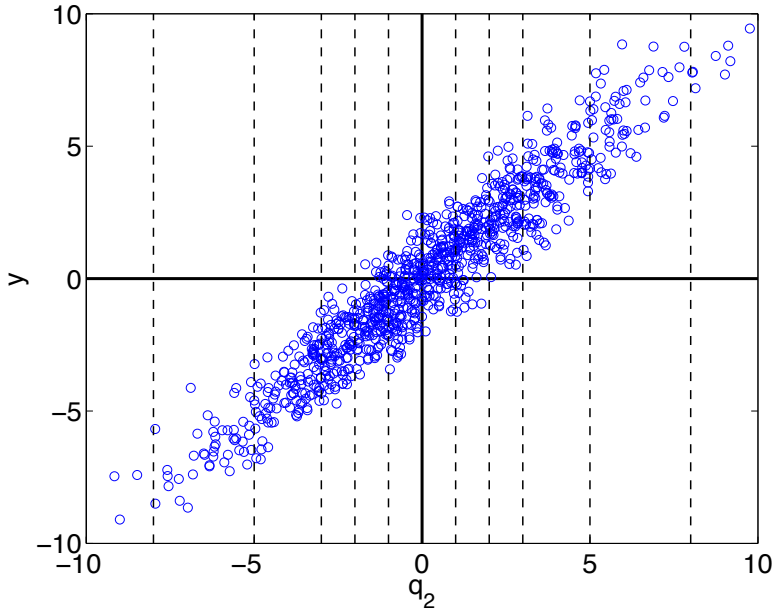
$$f(q) = f_0 + \sum_{i=1}^{p} f_i(q_i) + \sum_{1 \leqslant i < j \leqslant p} f_{ij}(q_i, q_j)$$

Analogy: Taylor or Fourier series

With appropriate assumptions,

$$f_0 = \int_\Gamma f(q)\,dq$$

$$f_i(q_i) = \int_{\Gamma^{p-1}} f(q)\,dq_{\sim i} - f_0$$



Variances:

$$D_i = \int_0^1 f_i^2(q_i)\,dq_i$$

$$D = \mathrm{var}(Y)$$

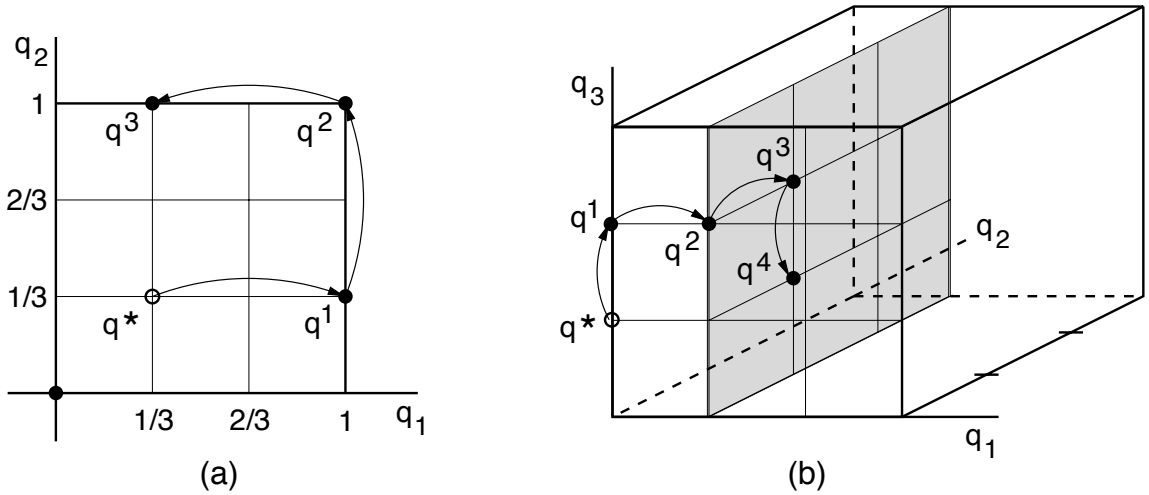
Sobol Indices: $S_i = \dfrac{D_i}{D}$

Statistical Interpretation:

$$D_i = \mathrm{var}[\mathbb{E}(Y|q_i)] \Rightarrow S_i = \frac{\mathrm{var}[\mathbb{E}(Y|q_i)]}{\mathrm{var}(Y)}$$

Morris Screening: Random Sampling of Approximated Derivatives

Example: Consider uniformly distributed parameters on $\Gamma = [0,1]^p$



Elementary Effect:

$$d_i^j = \frac{f(q^j + \Delta e_i) - F(q^j)}{\Delta} \quad , i^{th} \text{ parameter }, \; j^{th} \text{ sample}$$

Global Sensitivity Measures: r samples

$$\mu_i^* = \frac{1}{r} \sum_{j=1}^{r} |d_i^j(q)|$$

$$\sigma_i^2 = \frac{1}{r-1} \sum_{j=1}^{r} \left(d_i^j(q) - \mu_i \right)^2 \quad , \quad \mu_i = \frac{1}{r} \sum_{j=1}^{r} d_i^j(q)$$

SIR Disease Example

SIR Model:

$$\frac{dS}{dt} = \delta N - \delta S - \gamma k I S \quad , \; S(0) = S_0 \qquad \text{Susceptible}$$

$$\frac{dI}{dt} = \gamma k I S - (r + \delta) I \quad , \; I(0) = I_0 \qquad \text{Infectious}$$

$$\frac{dR}{dt} = rI - \delta R \quad , \; R(0) = R_0 \qquad \text{Recovered}$$

Note: Parameter set $q = [\gamma, k, r, \delta]$ is not identifiable

Assumed Parameter Distribution:

$$\gamma \sim \mathcal{U}(0,1) \; , \; k \sim Beta(\alpha, \beta) \; , \; r \sim \mathcal{U}(0,1) \; , \; \delta \sim \mathcal{U}(0,1)$$

Infection Coefficient	Interaction Coefficient	Recovery Rate	Birth/death Rate

Response:

$$y = \int_0^5 R(t, q) dt$$

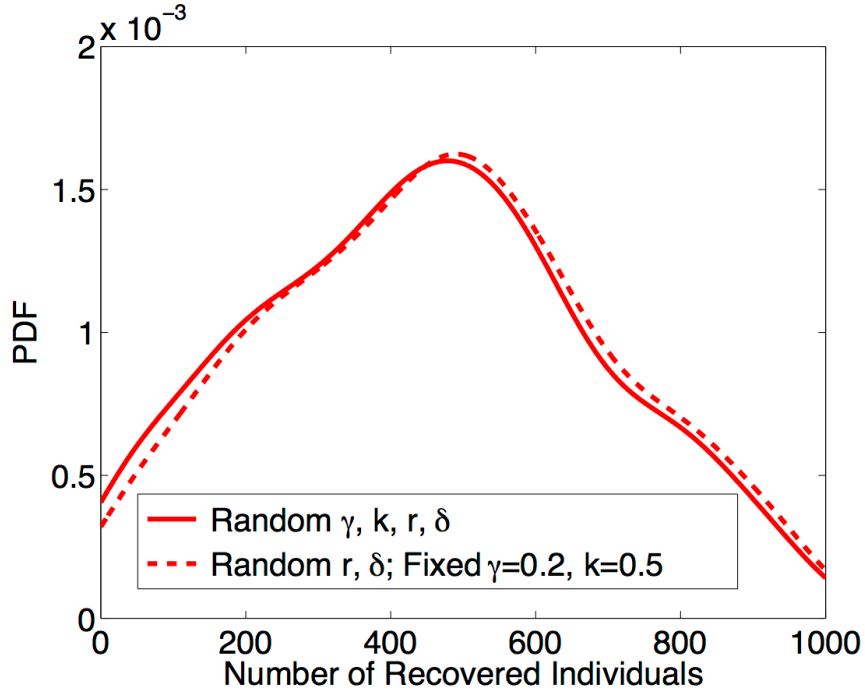
SIR Disease Example

Global Sensitivity Measures:

		γ	k	r	δ
Sobol	S_i	0.0997	0.0312	0.7901	0.1750
	S_{T_i}	−0.0637	−0.0541	0.5634	0.2029
Morris	μ_i^* $(\times 10^3)$	0.2532	0.2812	2.0184	1.2328
	σ_i $(\times 10^3)$	0.9539	1.6245	6.6748	3.9886

Influential Parameters

Result: Densities for $R(t_f)$ at $t_f = 5$



Note: Can fix non-influential parameters γ, k

Note: More during the project!

Concluding Remarks

Notes:

- UQ requires a synergy between engineering, statistics, and applied mathematics.
- Model calibration, model selection, uncertainty propagation and experimental design are natural in a Bayesian framework.
- Goal is to predict model responses with quantified and reduced uncertainties.
- Parameter selection is critical to isolate identifiable and influential parameters.
- Surrogate models critical for computationally intensive simulation codes.
- Codes and packages: Sandia Dakota, R, MATLAB, Python, nanoHUB.
- *Prediction is very difficult, especially if it's about the future*, Niels Bohr.