MA 574 – PROJECT 3

Due: Friday, March 29

(1) Consider a rod of length L with a fixed end at $x = 0$ as depicted in Figure 1. Let ρ, Y, c respectively denote the density, Young's modulus and internal damping coefficient and let $u(t, x)$ denote the longitudinal displacements. The rod decreases from a thickness $2h$ at $x = 0$ to a thickness of h at the free end and has uniform width b. Moreover, the rod has a mass m_L at $x = L$ along with a restraining spring with stiffness k_L that exerts a restoring force proportional to displacements $u(t, L)$.

(a) Balance forces to obtain a strong formulation of the model. Be sure to specify boundary conditions.

(b) Integrate by parts to obtain a corresponding weak formulation. Specify the space of test functions.

(c) Determine an appropriate basis and finite element solution to (b). Determine the resulting matrix vector system in second-order and first-order form. You should specify the components in your mass and stiffness matrices but you do not need to numerically evaluate the integrals.

(d) Consider the undamped rod, $c = 0$, with $m_L = k_L = 0$. Determine the kinetic and potential energies and use Hamiltonian (energy) principles to derive a weak formulation of the model. How does it compare with the model that you derived in (b)?

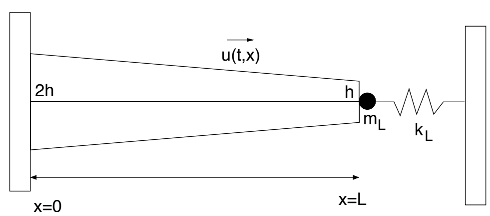

Figure 1: Geometry of the rod for Problem 1.

(2) Consider the weak formulation

$$\int_0^\ell \rho A \frac{\partial^2 u}{\partial t^2} \phi dx + \int_0^\ell \left[YA \frac{\partial u}{\partial x} + CA \frac{\partial^2 u}{\partial x \partial t} \right] \frac{d\phi}{dx} = - \left[k_\ell u(t, \ell) + c_\ell \frac{\partial u}{\partial t}(t, \ell) + m_\ell \frac{\partial^2 u}{\partial t^2}(t, \ell) \right] \phi(\ell),$$

which holds for all $(\phi, \phi(\ell)) \in V$. Using the linear hat functions defined in class, this yields the first-order system

$$\frac{dz}{dt} = Az(t)$$
$$z(0) = z_0.$$

Using the values $\rho = 2.7$ g/cm^3, $Y = 70$ GPa, $C = 10$ MPa, $A = 1$ cm^2, $k_\ell = 10$ MPa, $c_\ell = 10$ Pa, $m_\ell = 1$ g, and $\ell = 10$ cm, construct the matrix A and compute its eigenvalues. Do all of them have negative real part? Now construct the matrix A when the right hand side is positive and discuss the sign of the eigenvalues.