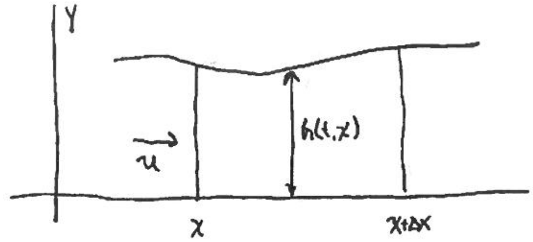
Problem 2

2a.) $\frac{d}{dt}\,\text{mass} = \frac{d}{dt}\int_x^{x+\Delta x} \bar{\rho}\, b\, h(t,x)\,dx$



Flux: $q = \bar{\rho}\, u(t,x)\, h(t,x)\, b$

Mass Balance:

$$\bar{\rho} b \int_x^{x+\Delta x} \frac{\partial h}{\partial t}\,dx = q(t,x) - q(t,x+\Delta x)$$

$$\Rightarrow \underline{\underline{\frac{\partial h}{\partial t} + \frac{\partial (uh)}{\partial x} = 0}} \quad \text{Continuity Equation}$$

b) $\int_x^{x+\Delta x} \frac{\partial}{\partial t}(\bar{\rho} u b h)\,dx = \left[\bar{\rho} u^2 h|_x - \bar{\rho} u^2 h|_{x+\Delta x}\right] b + \left[ph|_x - ph|_{x+\Delta x}\right] b$

$$\Rightarrow \underline{\underline{(\bar{\rho} h u)_t + (\bar{\rho} h u^2 + ph)_x = 0}} \quad \text{Momentum Equation}$$

c) Force at (t,x):

$$phb = \int_0^h b\bar{\rho} g y\,dy = \frac{1}{2}\bar{\rho} g h^2 b$$

Thus

$$\underline{\underline{(hu)_t + (hu^2 + \tfrac{1}{2} g h^2)_x = 0}}$$

d) Expanding derivatives yields

$$h_t u + h u_t + (hu)_x u + (hu) u_x + \tfrac{1}{2} g \cdot 2 h h_x = 0$$

$$\Rightarrow h\left[u_t + u u_x + g h_x\right] = 0$$

$$\Rightarrow u_t + \left(\tfrac{1}{2}u^2 + gh\right)_x = 0$$

System:

$$\underline{\underline{\begin{bmatrix} h \\ u \end{bmatrix}_t + \begin{bmatrix} uh \\ \frac{1}{2}u^2 + gh \end{bmatrix}_x = 0}}$$

e.) Boundary Conditions

$u|_{\text{body}} = 0$

$h|_{\text{bottom}} = 0$