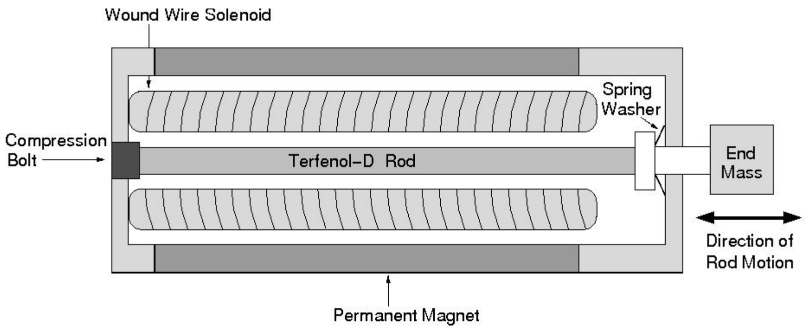
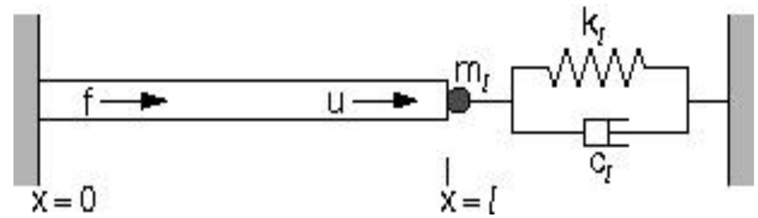
Development and Approximation of Rod Models

"He has Van Gogh's ear for music," Billy Wilder

Motivation: Terfenol-D Transducer





Questions:

- Are inputs uniform along rod length?
- How can this be determined?
- What magnetomechanical behavior must be incorporated in models?

Uniform Inputs: Spring Model

Spring Model: Consider magnetic field inputs H(t)

$$m\frac{d^2u}{dt^2} + c\frac{du}{dt} + ku = a_2 M^2(H(t))$$

$$u(t_0) = u_0$$

$$\frac{du}{dt}(t_0) = u_1$$

Questions:

- Can we compute an analytic solution?
- What numerical techniques can we use?
- How do we know if numerical techniques are converged?

Note:

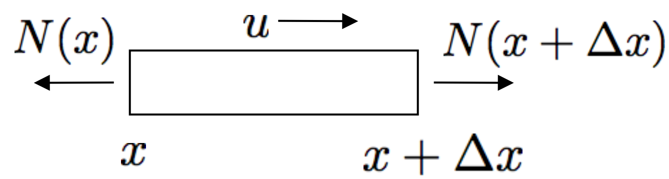
- Appropriate analytic and numerical techniques will depend, in part, on nature of M(H).

Nonuniform Inputs: Rod Model

Force Balance:

$$\int_x^{x+\Delta x} \rho A \frac{\partial^2 u}{\partial t^2}(t,s)ds = N(t,x+\Delta x) - N(t,x) + \int_x^{x+\Delta x} f(t,s)ds$$

Strategy: Multiply by $\frac{1}{\Delta x}$ and take limit to get



$$\rho A \frac{\partial^2 u}{\partial t^2} = \frac{\partial N}{\partial x} + f$$

Note: $N = \sigma A$, $\varepsilon = \lim_{\Delta x \to 0} \frac{\Delta u}{\Delta x} = \frac{\partial u}{\partial x}$

Constitutive Relations:

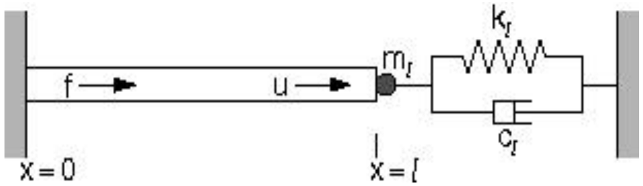
$$\sigma = Y\varepsilon + c\frac{\partial \varepsilon}{\partial t} - a_2 M^2(H)$$

$$M(H) = \int_0^\infty \int_{-\infty}^\infty \mu(H_c, H_I)\overline{M}(H + H_I; H_c)\, dH_I\, dH_c$$

Rod Model (Strong Formulation)

Rod Model:

$$\rho A \frac{\partial^2 u}{\partial t^2} - YA\frac{\partial^2 u}{\partial x^2} - cA\frac{\partial^3 u}{\partial x^2 \partial t} = f - a_2 A \frac{\partial (M^2(t))}{\partial x}$$



Boundary Conditions:

$$u(t,0) = 0$$

$$N(t,\ell) = -k_\ell u(t,\ell) - c_\ell \frac{\partial u}{\partial t}(t,\ell) - m_\ell \frac{\partial^2 u}{\partial t^2}(t,\ell)$$

Initial Conditions:

$$u(0,x) = u_0(x)$$

$$\frac{\partial u}{\partial t}(0,x) = u_1(x)$$

Issues:

Rod Model: Weak Formulation

State Space: $\xi(t) = (u(t,\cdot), u(t,\ell))$ in

$$X = L^2(0,\ell) \times \mathbb{R}$$

$$\langle \Phi_1, \Phi_2 \rangle_X = \int_0^\ell \rho A \phi_1 \phi_2 dx + m_\ell \varphi_1 \varphi_2$$

Space of Test Functions:

$$V = \left\{ \Phi = (\phi, \varphi) \in X \,|\, \phi \in H^1(0,\ell), \phi(0) = 0, \phi(\ell) = \varphi \right\}$$

$$\langle \Phi_1, \Phi_2 \rangle_V = \int_0^\ell YA \phi_1' \phi_2' dx + k_\ell \varphi_1 \varphi_2$$

Integration by Parts:

$$\int_0^\ell \rho A \frac{\partial^2 u}{\partial t^2} \phi dx + \int_0^\ell N \frac{d\phi}{dx} dx - N(t,\ell)\phi(\ell) = \int_0^\ell f \phi dx$$

Rod Model: Weak Formulation

Model:

$$\int_0^\ell \rho A \frac{\partial^2 u}{\partial t^2} \phi dx + \int_0^\ell \left[YA \frac{\partial u}{\partial x} + cA \frac{\partial^2 u}{\partial x \partial t} \right] \frac{d\phi}{dx} dx$$

$$= \int_0^\ell f \phi dx + a_2 A M^2(t) \int_0^\ell \frac{d\phi}{dx} dx$$

$$- \left[k_\ell u(t,\ell) + c_\ell \frac{\partial u}{\partial t}(t,\ell) + m_\ell \frac{\partial^2 u}{\partial t^2}(t,\ell) \right] \phi(\ell)$$

for all $(\phi, \varphi) = (\phi, \phi(\ell)) \in V$

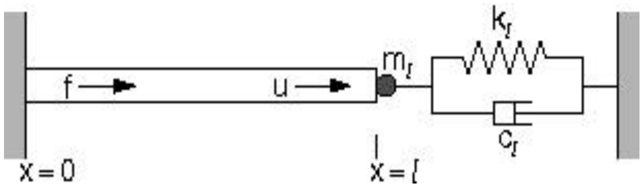
Note: For conservative case, weak formulation can also be derived using energy principles.

Rod Model: Strong and Weak Formulations

Rod Model:

$$\rho A \frac{\partial^2 u}{\partial t^2} - YA \frac{\partial^2 u}{\partial x^2} - cA \frac{\partial^3 u}{\partial x^2 \partial t} = f$$

Boundary Conditions:



$$u(t,0) = 0$$

$$N(t,\ell) = -k_\ell u(t,\ell) - c_\ell \frac{\partial u}{\partial t}(t,\ell) - m_\ell \frac{\partial^2 u}{\partial t^2}(t,\ell)$$

Weak Formulation:

$$\int_0^\ell \rho A \frac{\partial^2 u}{\partial t^2} \phi dx + \int_0^\ell \left[YA \frac{\partial u}{\partial x} + cA \frac{\partial^2 u}{\partial x \partial t} \right] \frac{d\phi}{dx} dx$$

$$= \int_0^\ell f \phi dx - \left[k_\ell u(t,\ell) + c_\ell \frac{\partial u}{\partial t}(t,\ell) + m_\ell \frac{\partial^2 u}{\partial t^2}(t,\ell) \right] \phi(\ell)$$

for all $(\phi, \varphi) = (\phi, \phi(\ell)) \in V$

Weak Formulation: Energy Principles

Issues: Need conservative problem; take $c = f = k_\ell = c_\ell = m_\ell = 0$

Kinetic Energy: $K = \frac{1}{2}\int_0^L \rho A u_t^2(t,x)dx$

Potential Energy: $U = \frac{1}{2}\int_0^L N\varepsilon dx = \frac{1}{2}\int_0^L \sigma A\varepsilon dx = \frac{1}{2}\int_0^L YAu_x^2(t,x)dx$

Weak Formulation: Energy Principles

'Action' Integral:

$$\mathcal{A}[u] = \int_{t_0}^{t_1} \mathcal{L} dt$$

where

$$\mathcal{L} = \frac{1}{2} \int_0^L \left[\rho A u_2^2 - Y A u_x^2\right] dx$$

Hamilton's Principle:

$$\delta \mathcal{A}[u; \Phi] = \frac{d}{d\varepsilon} \mathcal{A}[u + \varepsilon \Phi]\Big|_{\varepsilon=0} = 0$$

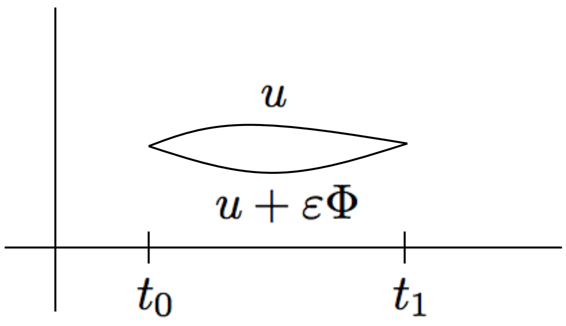
Admissible variations:

$$\widehat{u}(t,x) = u(t,x) + \varepsilon \eta(t) \phi(x)$$

where

(i) $\eta(t_0) = \eta(t_1) = 0$

(ii) $\phi \in V = H_0^1(0, L)$



Weak Formulation: Energy Principles

Hamilton's Principle:

$$\frac{d}{d\varepsilon}\mathcal{A}[u+\varepsilon\Phi]\Big|_{\varepsilon=0} = \frac{1}{2}\frac{d}{d\varepsilon}\int_{t_0}^{t_1}\int_0^L \left[\rho A(u_t+\varepsilon\dot{\eta}\phi)^2 - YA(u_x+\varepsilon\eta\phi')^2\right]dxdt\Big|_{\varepsilon=0}$$

$$= \int_{t_0}^{t_1}\int_0^L \left[\rho A(u_t+\varepsilon\dot{\eta}\phi)\dot{\eta}\phi - YA(u_x+\varepsilon\eta\phi')\eta\phi'\right]dxdt\Big|_{\varepsilon=0}$$

$$= \int_{t_0}^{t_1}\int_0^L \left[\rho A u_t\dot{\eta}\phi - YA u_x\eta\phi'\right]dxdt$$

$$= -\int_{t_0}^{t_1}\eta(t)\int_0^L \left[\rho A u_{tt}\phi + YA u_x\phi'\right]dxdt$$

Weak Formulation:

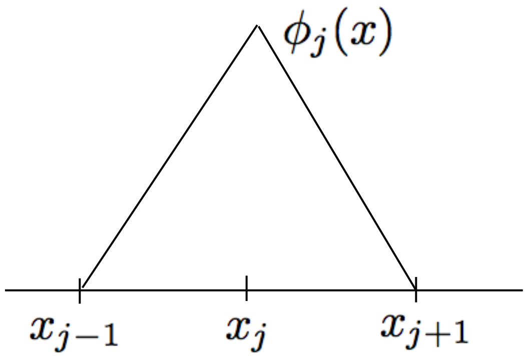
$$\int_0^L \rho A u_{tt}\phi dx + YA\int_0^L u_x\phi' dx = 0 \quad \text{for all } \phi\in V$$

Note: Use integration by parts to show equivalence to strong formulation

Approximation Techniques for the Rod Model: Galerkin

Linear Basis:

$$\phi_j(x) = \frac{1}{h}\begin{cases} x - x_{j-1}\,, & x_{j-1} \le x < x_j \\ x_{j+1} - x\,, & x_j \le x \le x_{j+1} \\ 0\,, & \text{otherwise} \end{cases}$$



Approximate Solution:

$$u^N(t,x) = \sum_{j=1}^{N} u_j(t)\phi_j(x)$$

System:

$$\sum_{j=1}^{N} \ddot{u}_j(t)\int_0^\ell \rho A\phi_i\phi_j dx + \sum_{j=1}^{N} \dot{u}_j(t)\int_0^\ell cA\phi_i'\phi_j' dx + \sum_{j=1}^{N} u_j(t)\int_0^\ell YA\phi_i'\phi_j' dx$$

$$= \int_0^\ell f\phi_i dx - \Big(k_\ell u_N(t)\phi_N(\ell) + c_\ell \dot{u}_N(t)\phi_N(\ell) + m_\ell \ddot{u}_N(t)\phi_N(\ell)\Big)\phi_N(\ell)$$

for $i = 1, \cdots N$

Approximation Techniques for the Rod Model: Galerkin

2nd-Order Vector System:

$$\mathbb{M}\,\ddot{\mathbf{u}}(t) + \mathbb{Q}\,\dot{\mathbf{u}} + \mathbb{K}\,\mathbf{u}(t) = \mathbf{f}(t)$$

where $\mathbf{u}(t) = [u_1(t), \ldots, u_N(t)]^T$ and

$$[\mathbb{M}]_{ij} = \begin{cases} \displaystyle\int_0^\ell \rho A \phi_i \phi_j \, dx & , \quad i \neq N \text{ or } j \neq N \\ \displaystyle\int_0^\ell \rho A \phi_i \phi_j \, dx + m_\ell & , \quad i = N \text{ and } j = N \end{cases}$$

$$[\mathbb{K}]_{ij} = \begin{cases} \displaystyle\int_0^\ell Y A \phi_i' \phi_j' \, dx & , \quad i \neq N \text{ or } j \neq N \\ \displaystyle\int_0^\ell Y A \phi_i' \phi_j' \, dx + k_\ell & , \quad i = N \text{ and } j = N \end{cases}$$

$$[\mathbb{Q}]_{ij} = \begin{cases} \displaystyle\int_0^\ell c A \phi_i' \phi_j' \, dx & , \quad i \neq N \text{ or } j \neq N \\ \displaystyle\int_0^\ell c A \phi_i' \phi_j' \, dx + c_\ell & , \quad i = N \text{ and } j = N \end{cases}$$

$$[\mathbf{f}]_i = \int_0^\ell f \phi_i dx$$

Approximation Techniques for the Rod Model: Galerkin

Matrices: Constant coefficients

$$\mathbb{M} = \rho A h \begin{bmatrix} \frac{2}{3} & \frac{1}{6} & 0 & \cdots & & 0 \\ \frac{1}{6} & \frac{2}{3} & \frac{1}{6} & & & \\ & \ddots & \ddots & \ddots & & \\ & & \frac{1}{6} & \frac{2}{3} & & \frac{1}{6} \\ 0 & \cdots & 0 & \frac{1}{6} & \frac{1}{3} + \frac{m_\ell}{h} \end{bmatrix} \quad , \quad \mathbb{K} = \frac{YA}{h} \begin{bmatrix} 2 & -1 & 0 & \cdots & & 0 \\ -1 & 2 & -1 & & & \\ & \ddots & \ddots & \ddots & & \\ & & -1 & 2 & & -1 \\ 0 & \cdots & & 0 & -1 & 1 + hk_\ell \end{bmatrix}$$

First-Order System:

$$\dot{\mathbf{z}}(t) = \mathbb{A}\,\mathbf{z}(t) + \mathbf{F}(t)$$

$$\mathbf{z}(0) = \mathbf{z}_0$$

where

$$\mathbb{A} = \begin{bmatrix} 0 & \mathbb{I} \\ -\mathbb{M}^{-1}\mathbb{K} & -\mathbb{M}^{-1}\mathbb{Q} \end{bmatrix} \quad , \quad \mathbf{F}(t) = \begin{bmatrix} 0 \\ \mathbb{M}^{-1}\mathbf{f}(t) \end{bmatrix}$$

Note: Codes available at http://www4.ncsu.edu/~rsmith/Smart_Material_Systems/Chapter8/

Approximation Techniques for the Rod Model: Finite Element

Motivating Problem: $\rho A \frac{\partial^2 u}{\partial t^2} + YA \frac{\partial^2 u}{\partial x^2} = 0$

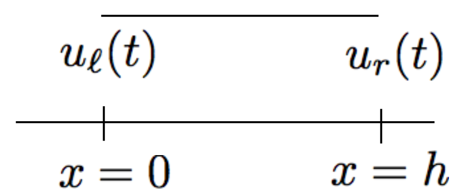
Local Basis Elements: Take

$$\begin{aligned} u(t,x) &= a_0(t) + a_1(t)x \\ &= \boldsymbol{\varphi}^T(x)\mathbf{a}(t) \end{aligned}$$

where $\mathbf{a}(t) = [a_0(t)\,,\, a_1(t)]^T$ and $\boldsymbol{\varphi}(x) = [1\,,\, x]^T$. Then

$$\begin{bmatrix} u_\ell(t) \\ u_r(t) \end{bmatrix} = \begin{bmatrix} 1 & 0 \\ 1 & h \end{bmatrix} \begin{bmatrix} a_0(t) \\ a_1(t) \end{bmatrix}$$

$$\Rightarrow \mathbf{u}(t) = \mathbb{T}\mathbf{a}(t)$$



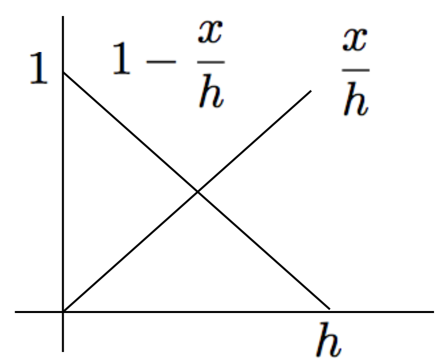
By observing that $\mathbf{a}(t) = \mathbb{S}\mathbf{u}(t)$, where

$$\mathbb{S} = \mathbb{T}^{-1} = \begin{bmatrix} 1 & 0 \\ -\frac{1}{h} & \frac{1}{h} \end{bmatrix},$$

it follows that displacements can be represented as

$$u(t,x) = \boldsymbol{\phi}^T(x)\mathbf{u}(t)$$

where $\phi^T(x) = [1 - \frac{x}{h}\,,\, \frac{x}{h}]$.



Approximation Techniques for the Rod Model: Finite Element

`Action' Integral:

$$\mathcal{A} = \int_{t_0}^{t_1} [K - U] dt = \int_{t_0}^{t_1} \frac{1}{2} \int_0^h \left[\rho A u_t^2(t,x) - Y A u_x^2(t,x)\right] dx dt$$

K: Kinetic Energy

U: Potential Energy

Here

$$u_x^2(t,x) = \mathbf{u}^T(t)\mathbb{S}^T\mathbb{D}(x)\mathbb{S}\mathbf{u}(t) \ , \ u_t^2(t,x) = \dot{\mathbf{u}}^T(t)\mathbb{S}^T\mathbb{F}(x)\mathbb{S}\dot{\mathbf{u}}(t)$$

where

$$\mathbb{D}(x) = \boldsymbol{\varphi}_x(x)\boldsymbol{\varphi}_x^T(x) = \begin{bmatrix} 0 & 0 \\ 0 & 1 \end{bmatrix} \quad , \quad \mathbb{F}(x) = \boldsymbol{\varphi}(x)\boldsymbol{\varphi}^T(x) = \begin{bmatrix} 1 & x \\ x & x^2 \end{bmatrix}$$

Thus

$$\mathcal{A}[\mathcal{U}] = \int_{t_0}^{t_1} \frac{1}{2} \left[\dot{\mathbf{u}}^T(t)\mathbb{M}_e\dot{\mathbf{u}}(t) - \mathbf{u}^T(t)\mathbb{K}_e\mathbf{u}(t)\right] dt$$

where $\mathcal{U} = (\mathbf{u}, \dot{\mathbf{u}})$ and

$$\mathbb{M}_e = \rho A \mathbb{S}^T \cdot \int_0^h \mathbb{F}(x) dx \cdot \mathbb{S} \quad , \quad \mathbb{K}_e = Y A \mathbb{S}^T \cdot \int_0^h \mathbb{D}(x) dx \cdot \mathbb{S}$$

$$\Rightarrow \mathbb{M}_e = \rho A h \begin{bmatrix} \frac{1}{3} & \frac{1}{6} \\ \frac{1}{6} & \frac{1}{3} \end{bmatrix} \quad , \quad \mathbb{K}_e = \frac{YA}{h} \begin{bmatrix} 1 & -1 \\ -1 & 1 \end{bmatrix}$$

M_e: *Local* Mass Matrix

K_e: *Local* Stiffness Matrix

Approximation Techniques for the Rod Model: Finite Element

Hamilton's Principle: See "String and Membrane" lectures

$$\frac{d}{d\varepsilon}\mathcal{A}[\mathcal{U}+\varepsilon\Upsilon]\bigg|_{\varepsilon=0} = 0$$

where $\Upsilon = (\boldsymbol{\eta}, \dot{\boldsymbol{\eta}})$ and it is assumed that $\boldsymbol{\eta}(t_0) = \boldsymbol{\eta}(t_1) = 0$. Here

$$\mathcal{A}[\mathcal{U}+\varepsilon\Upsilon] = \int_{t_0}^{t_1} \frac{1}{2}\left[(\dot{\mathbf{u}}+\varepsilon\dot{\boldsymbol{\eta}})^T \mathbb{M}_e (\dot{\mathbf{u}}+\varepsilon\dot{\boldsymbol{\eta}}) - (\mathbf{u}+\varepsilon\boldsymbol{\eta})^T \mathbb{K}_e (\mathbf{u}+\varepsilon\boldsymbol{\eta})\right] dt$$

$$\Rightarrow \frac{d}{d\varepsilon}\mathcal{A}[\mathcal{U}+\varepsilon\Upsilon] = \int_{t_0}^{t_1} \frac{1}{2}\Big[\dot{\boldsymbol{\eta}}^T\mathbb{M}_e(\dot{\mathbf{u}}+\varepsilon\dot{\boldsymbol{\eta}}) + (\dot{\mathbf{u}}+\varepsilon\dot{\boldsymbol{\eta}})^T\mathbb{M}_e\dot{\boldsymbol{\eta}}$$
$$-\boldsymbol{\eta}^T\mathbb{K}_e(\mathbf{u}+\varepsilon\boldsymbol{\eta}) + (\mathbf{u}+\varepsilon\boldsymbol{\eta})^T\mathbb{K}_e\boldsymbol{\eta}\Big] dt$$

$$\Rightarrow \frac{d}{d\varepsilon}\mathcal{A}[\mathcal{U}+\varepsilon\Upsilon]\bigg|_{\varepsilon=0} = \int_{t_0}^{t_1}\left[\dot{\boldsymbol{\eta}}^T\mathbb{M}_e\dot{\mathbf{u}} - \boldsymbol{\eta}^T\mathbb{K}_e\mathbf{u}\right] dt \quad \text{(since } \dot{\mathbf{u}}^T\mathbb{M}_e\dot{\boldsymbol{\eta}} = \dot{\boldsymbol{\eta}}^T\mathbb{M}_e\dot{\mathbf{u}}\text{)}$$
$$= -\int_{t_0}^{t_1}\boldsymbol{\eta}\left[\mathbb{M}_e\ddot{\mathbf{u}} + \mathbb{K}_e\mathbf{u}\right] dt \qquad \text{(integration by parts)}$$

Local System:

$$\mathbb{M}_e\ddot{\mathbf{u}}(t) + \mathbb{K}_e\mathbf{u} = 0$$

Approximation Techniques for the Rod Model: Finite Element

Global Matrices: Consider two subregions so $h = \frac{\ell}{2}$. Here

$$\mathbf{u}(t) = [u_{1\ell}(t), u_{2\ell}(t), u_{2r}(t)]^T$$

and

$$\mathbb{M}\,\ddot{\mathbf{u}} + \mathbb{K}\,\mathbf{u} = 0$$

where

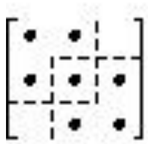
$$\mathbb{M} = \rho A h \begin{bmatrix} \frac{1}{3} & \frac{1}{6} & 0 \\ \frac{1}{6} & \frac{2}{3} & \frac{1}{6} \\ 0 & \frac{1}{6} & \frac{1}{3} \end{bmatrix} \quad , \quad \mathbb{K} = \frac{YA}{h} \begin{bmatrix} 1 & -1 & 0 \\ -1 & 2 & -1 \\ 0 & -1 & 1 \end{bmatrix}.$$

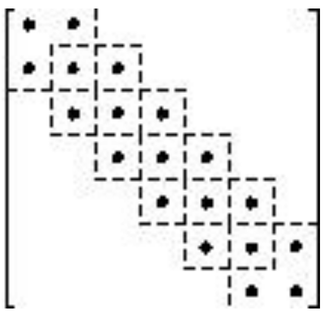
Note: Second row obtained by summing

$$\rho A h \left(\frac{1}{6}\ddot{u}_{1\ell} + \frac{1}{3}\ddot{u}_{2\ell} \right) + \frac{YA}{h}\left(-u_{1\ell} + u_{2\ell}\right) = 0$$

$$\rho A h \left(\frac{1}{3}\ddot{u}_{2\ell} + \frac{1}{6}\ddot{u}_{2r} \right) + \frac{YA}{h}\left(u_{2\ell} - u_{2r}\right) = 0$$

after enforcing $u_{1r} = u_{2\ell}$ and $\ddot{u}_{1r} = \ddot{u}_{2\ell}$.





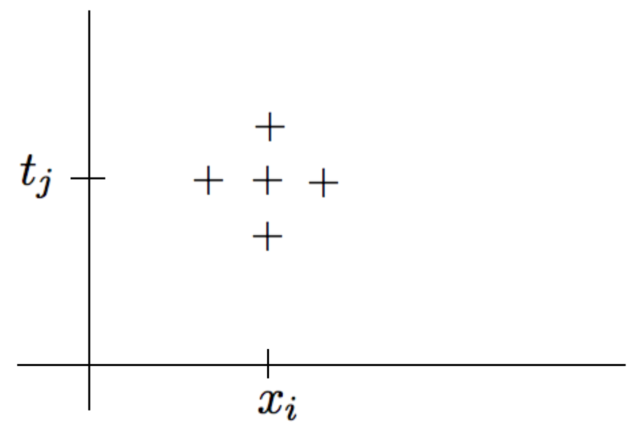
Finite Difference Techniques for the Rod Model

Consider first

$$u_{tt} - K^2 u_{xx} = F$$

$$u(t,0) = u(t,\ell) = 0$$

$$u(0,x) = f(x)\ ,\ u_t(0,x) = g(x)$$



Grid: $\Delta = \{(x_i,t_j)\,|\,x_i = ih, t_j = jk\}$

System: $m = K\dfrac{k}{h}$

$$\frac{1}{k^2}\left[u_{i,j-1} - 2u_{i,j} + u_{i,j+1}\right] - \frac{K^2}{h^2}\left[u_{i-1,j} - 2u_{i,j} + u_{i+1,j}\right] = F(t_j,x_i)$$

$$\Rightarrow u_{i,j+1} = m^2\left[u_{i+1,j} + u_{i-1,j}\right] + 2(1-m^2)u_{i,j} - u_{i,j-1} + k^2 F(t_j,x_i)$$

Initial Conditions:

$$g(x_i) \approx \frac{u_{i,1} - u_{i,-1}}{2k} \Rightarrow u_{i,-1} = u_{i,1} - 2kg(x_k)$$

$$\Rightarrow u_{i,1} = \frac{m^2}{2}\left[f(x_{i+1}) + f(x_{i-1})\right] + (1-m^2)f(x_i) + kg(x_i) + \frac{k^2}{2}F(t_0,x_i)$$

Finite Difference Techniques for the Rod Model

Now consider

$$u_{tt} - c^2 u_{xxt} - K^2 u_{xx} = F$$

$$u(t,0) = u(t,\ell) = 0$$

$$u(0,x) = f(x) \ , \ u_t(0,x) = g(x)$$

Question: What is finite difference scheme?