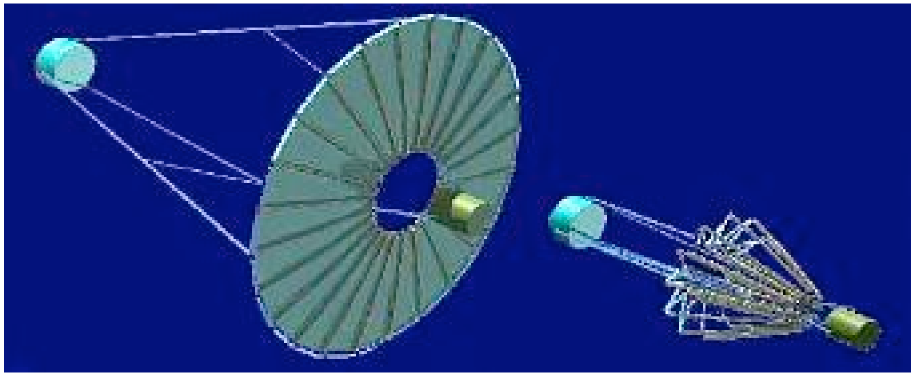
String and Membrane Models

"He has all the virtues I dislike and none of the vices I admire” Winston Churchill

Applications

Membrane Mirrors:

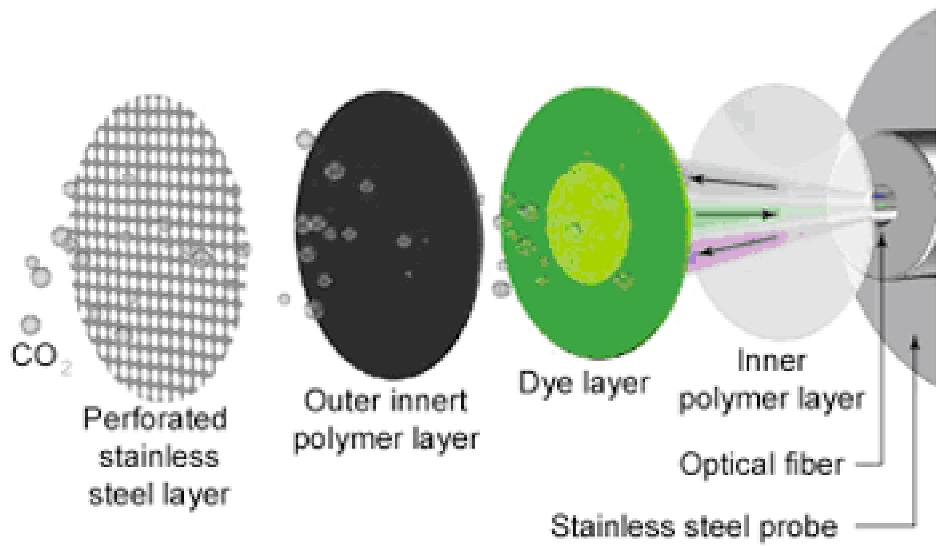
- E.g., Kirtland AFB
- Issue: How do you model and compensate for wrinkles?



Applications

Chemical and Biological Sensors:

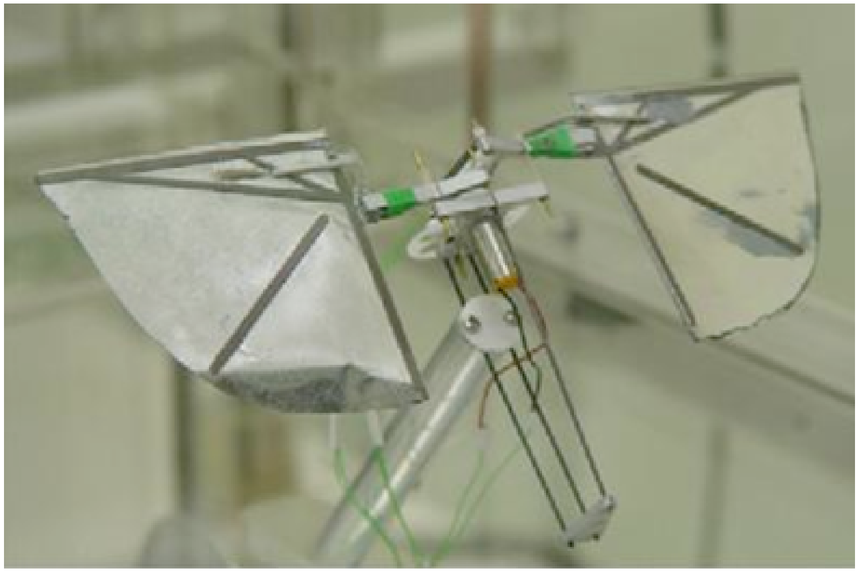
- Often involve ionic polymers

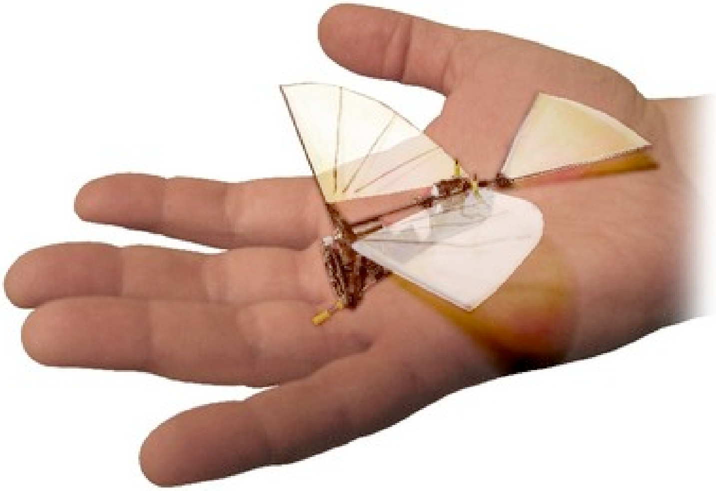


Applications

Active Polymers:

- e.g., PVDF (polyvinylidene fluoride) for MAV
- Structural sensors





Applications

Strings or Wires:

- 1-D version of membranes
- e.g., transmission line design to minimize ice or wind failure

String Model

Assumptions:

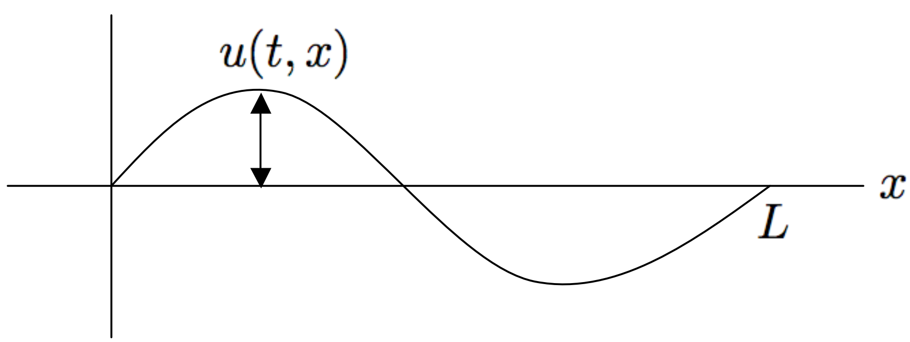
- Small displacements permit linearization
- Density $\rho_0(x)$: Linear density at equilibrium (mass/unit length)

  $\rho(t,x)$: Linear density at time t (mass/unit length)

  Conservation of Mass: Consider arbitrary interval $[x_1, x_2]$

$$m = \int_{x_1}^{x_2} \rho_0(x)dx = \int_{x_1}^{x_2} \rho(t,x)\sqrt{1+u_x^2(t,x)}dx$$

$$\Rightarrow \rho_0 = \rho\sqrt{1+u_x^2}$$



String Model -- Strong Formulation

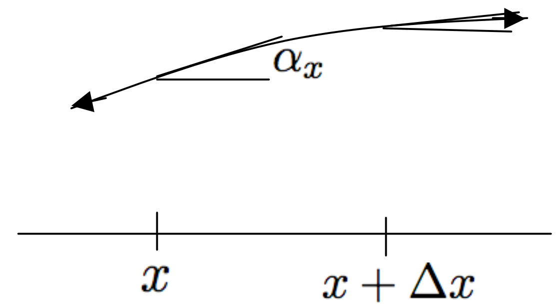
Strong Formulation: Based on force (momentum) balance

Horizontal Forces: Tension T_x is force in string at x

$$T_{x+\Delta x} \cos \alpha_{x+\Delta x} - T_x \cos \alpha_x = 0$$

$$\Rightarrow \frac{\partial}{\partial x}(T_x \cos \alpha_x) = 0$$

$$\Rightarrow T_x \cos \alpha_x = T(t) > 0$$



Conclusion: Tension $T(t)$ is constant in space

Vertical Forces:

1. Vertical component of tension

$$\begin{aligned} T_{x+\Delta x} \sin \alpha_{x+\Delta x} - T_x \sin \alpha_x &= T\left[\frac{\sin \alpha_{x+\Delta x}}{\cos \alpha_{x+\Delta x}} - \frac{\sin \alpha_x}{\cos \alpha_x}\right] \\ &\approx T\left[u_x(t, x+\Delta x) - u_x(t,x)\right] \end{aligned}$$

String Model -- Strong Formulation

Vertical Forces:

2. Weight

$$-\int \rho g ds = -\int_x^{x_\Delta x} \rho g \sqrt{1+u_x^2} dx = -\int_x^{x+\Delta x} \rho_0 g dx$$

3. External load; e.g., bowing of a violin string

$$\int \rho f ds = \int_x^{x+\Delta x} \rho_0 f(t,x) dx$$

4. Viscous forces: e.g., string in water, internal damping

$$-\int \rho k u_t ds = -\int_x^{x+\Delta x} \rho_0(x) k u_t(t,x) dx$$

Newton's 2nd Law:

$$\int_x^{x+\Delta x} \rho_0 u_{tt} dx = T\left[u_x(t,x+\Delta x) - u_x(t,x)\right]$$
$$-\int_x^{x+\Delta x} \rho_0(x) k u_t(t,x) dx + \int_x^{x+\Delta x} \rho_0 (f-g) dx$$

$$\Rightarrow \rho_0 u_{tt} + k\rho_0 u_t = T u_{xx} + \rho_0 (f-g)$$

String Model -- Strong Formulation

Model: If ρ_0 is constant, take $c^2 = T/\rho_0$ and $F = f - g$ to get

$$u_{tt} + ku_t = c^2 u_{xx} + F$$

$$u(0, x) = u_0(x) \ , \ u_t(0, x) = u_1(x)$$

$$u(t, 0) = u(t, L) = 0$$

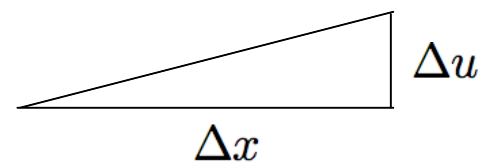
Kinetic and Potential Energy

Kinetic Energy:

$$K = \frac{1}{2}\int_0^L \rho_0 u_t^2(t,x)\,dx$$

Potential Energy: Find change in length and multiply by T

$$\begin{aligned}\Delta L &= \sqrt{(\Delta x)^2 + (\Delta u)^2} - (\Delta x) \\ &= \Delta x \left[\sqrt{1 + \left(\frac{\Delta u}{\Delta x}\right)^2} - 1\right] \\ &\approx \frac{1}{2}\left(\frac{\Delta u}{\Delta x}\right)^2 \Delta x \qquad \text{since } \sqrt{1+x} \approx 1 + \frac{1}{2}x\end{aligned}$$



or

$$\int_0^L \sqrt{1+u_x^2}\,dx - L \approx \frac{1}{2}\int_0^L u_x^2\,dx$$

Thus

$$U = \frac{1}{2}\int_0^L T u_x^2(t,x)\,dx$$

String Model -- Weak Formulation

‘Action’ Integral:

$$\mathcal{A}[u] = \int_{t_0}^{t_1} \mathcal{L} dt$$

where

$$\mathcal{L} = \frac{1}{2} \int_0^L \left[\rho_0 u_t^2 - T u_x^2\right] dx$$

Hamilton’s Principle:

$$\delta \mathcal{A}[u; \Phi] = \frac{d}{d\varepsilon} \mathcal{A}[u + \varepsilon \Phi]\Big|_{\varepsilon} = 0$$

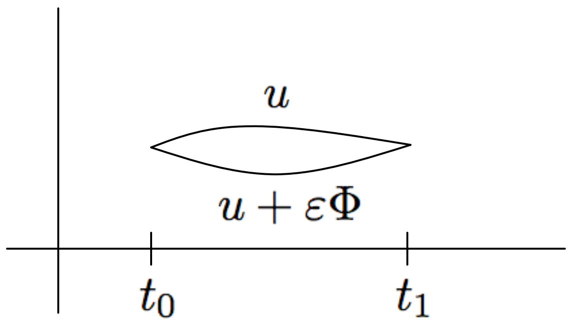
Admissible variations:

$$\widehat{u}(t,x) = u(t,x) + \varepsilon \eta(t) \phi(x)$$

where

(i) $\eta(t_0) = \eta(t_1) = 0$

(ii) $\phi \in V = H_0^1(0, L)$



String Model -- Weak Formulation

Hamilton's Principle:

$$
\begin{aligned}
\frac{d}{d\varepsilon}\mathcal{A}[u+\varepsilon\Phi]\Big|_{\varepsilon} &= \frac{1}{2}\frac{d}{d\varepsilon}\int_{t_0}^{t_1}\int_0^L \left[\rho_0(u_t+\varepsilon\dot{\eta}\phi)^2 - T(u_x+\varepsilon\eta\phi')^2\right] dxdt\Big|_{\varepsilon=0} \\
&= \int_{t_0}^{t_1}\int_0^L \left[\rho_0(u_t+\varepsilon\dot{\eta}\phi)\dot{\eta}\phi - T(u_x+\varepsilon\eta\phi')\eta\phi'\right] dxdt\Big|_{\varepsilon=0} \\
&= \int_{t_0}^{t_1}\int_0^L \left[\rho_0 u_t\dot{\eta}\phi - Tu_x\eta\phi'\right] dxdt \\
&= -\int_{t_0}^{t_1}\eta(t)\int_0^L \left[\rho_0 u_{tt}\phi + Tu_x\phi'\right] dxdt
\end{aligned}
$$

Weak Formulation:

$$
\int_0^L \rho_0 u_{tt}\phi dx + T\int_0^L u_x\phi' dx = 0 \quad \text{for all } \phi \in V
$$

Note: Use integration by parts to show equivalence to strong formulation

String Model -- Solution Techniques

Numerical Methods: See acoustic lectures

Analytic Solutions: See acoustic lectures

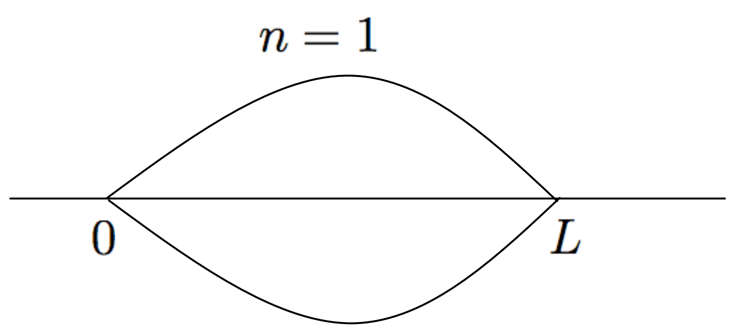
$$u(t,x) = \sum_{n=1}^{\infty} [a_n \cos(\lambda_n ct) + b_n \sin(\lambda_n ct)] \sin(\lambda_n x)$$

where

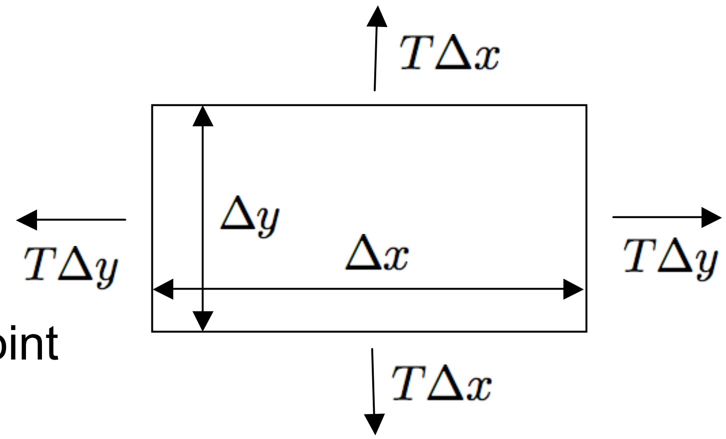
$$\lambda_n = \frac{n\pi}{L}$$

Recall: $\lambda = \frac{c}{f}$

Note: $f_n = \frac{n}{2L}\sqrt{T/\rho_0}$



Membrane Model



Assumptions:

- Small displacements permit linearization
- Density: ρ_0 (kg/m^2)
- Assume tensile stress T is same at every point

Force due to Tensions $T\Delta y$:

$$T\Delta y\left[u_x(t,x+\Delta x)-u_x(t,x)\right]=T\Delta y\Delta x\left[\frac{u_x(t,x+\Delta x)-u_x(t,x)}{\Delta x}\right]$$

Force due to Tensions $T\Delta x$: Similar

Force Balance:

$$\int_x^{x+\Delta x}\int_y^{y+\Delta y}\rho_0 u_{tt}\,dxdy=T\Delta x\Delta y\left[\frac{u_x(t,x+\Delta x,y)-u_x(t,x,y)}{\Delta x}\right]$$

$$+T\Delta x\Delta y\left[\frac{u_y(t,x,y+\Delta y)-u_y(t,x,y)}{\Delta y}\right]$$

Membrane Model

Membrane Model:

$$\rho_0 \frac{\partial^2 u}{\partial t^2} = T \left[\frac{\partial^2 u}{\partial x^2} + \frac{\partial^2 u}{\partial y^2} \right]$$