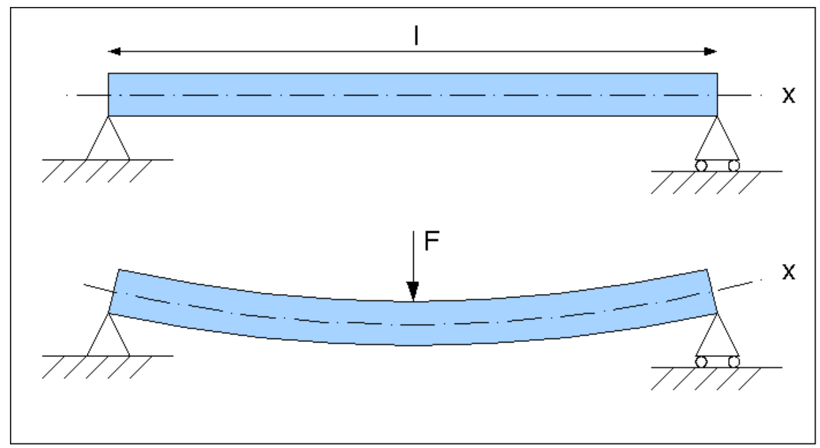
Fluids Models

“Water is fluid, soft and yielding. But water will wear away rock, which is rigid and cannot yield. As a rule, whatever is fluid, soft and yielding will overcome whatever is rigid and hard. This is another paradox: what is soft is strong” Lao-Tzu

Fluid Phenomena

Definition (Fluid): Any liquid or gas that cannot sustain a shearing force when at rest and that undergoes a continuous change in shape when subjected to such a stress. Compressed fluids exert an outward pressure that is perpendicular to the walls of their containers. A perfect fluid lacks viscosity, but real fluids do not.

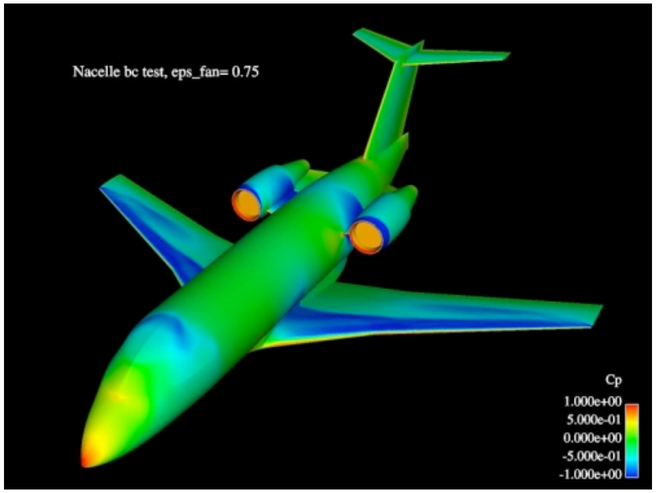
Comparison: A solid deforms only until the external and internal forces are balanced.

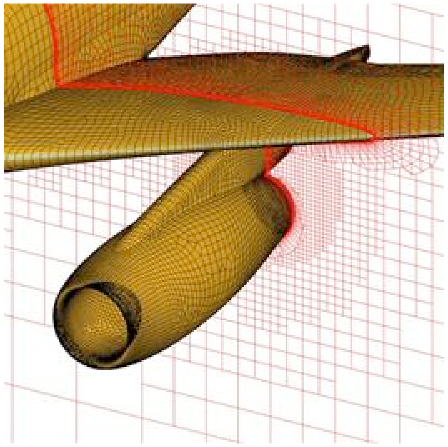


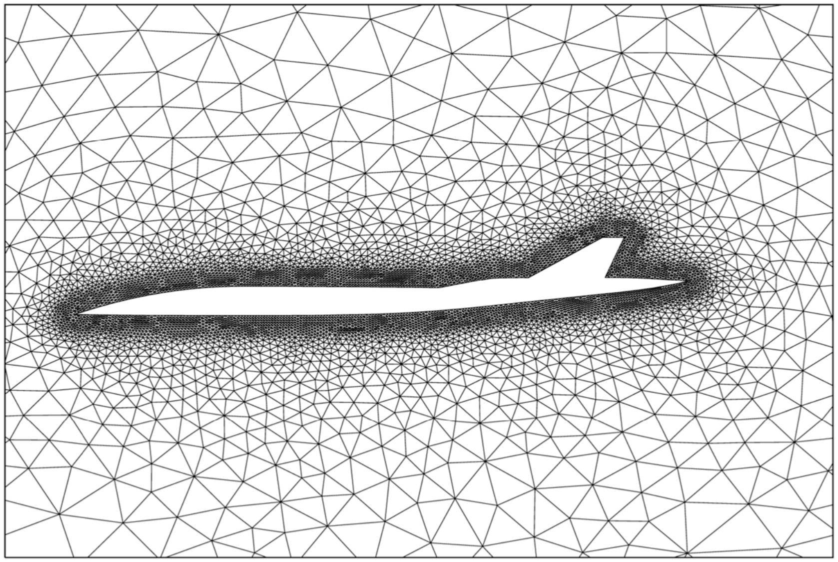
Chimera: Granular materials (e.g., sand)

Research Topics: Improved Aeronautic Designs

Aircraft Design: Computational fluid dynamics (CFD) codes widely used in the design of the Boeing 787.

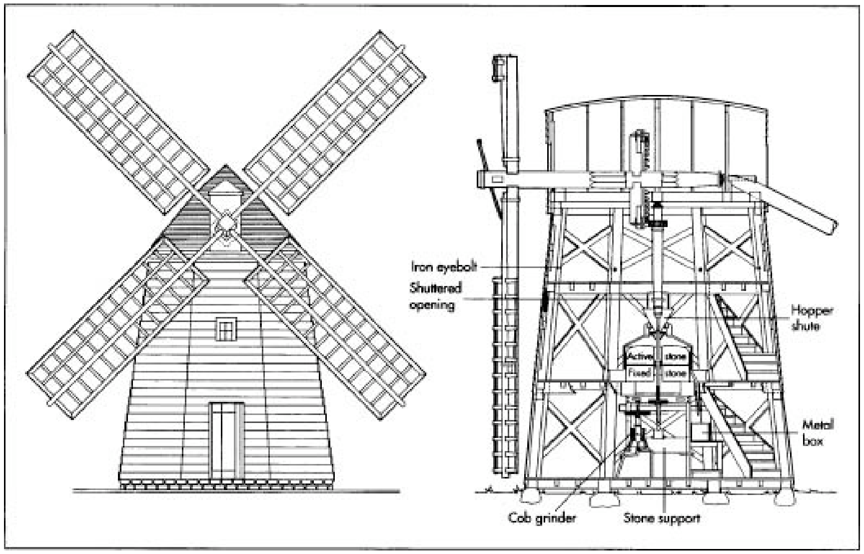






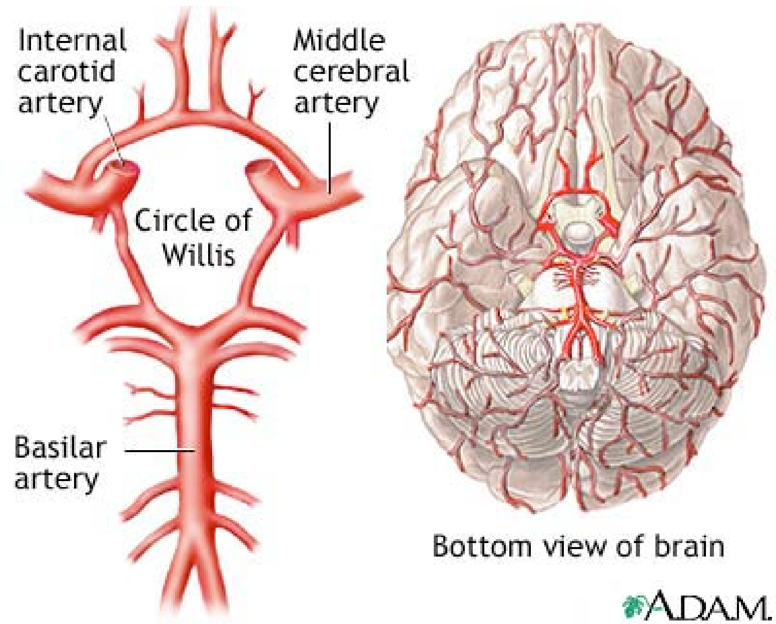
Research Topics: Improved Aeronautic Designs

Windmill Design: Improved airfoils yield substantially improved efficiency.

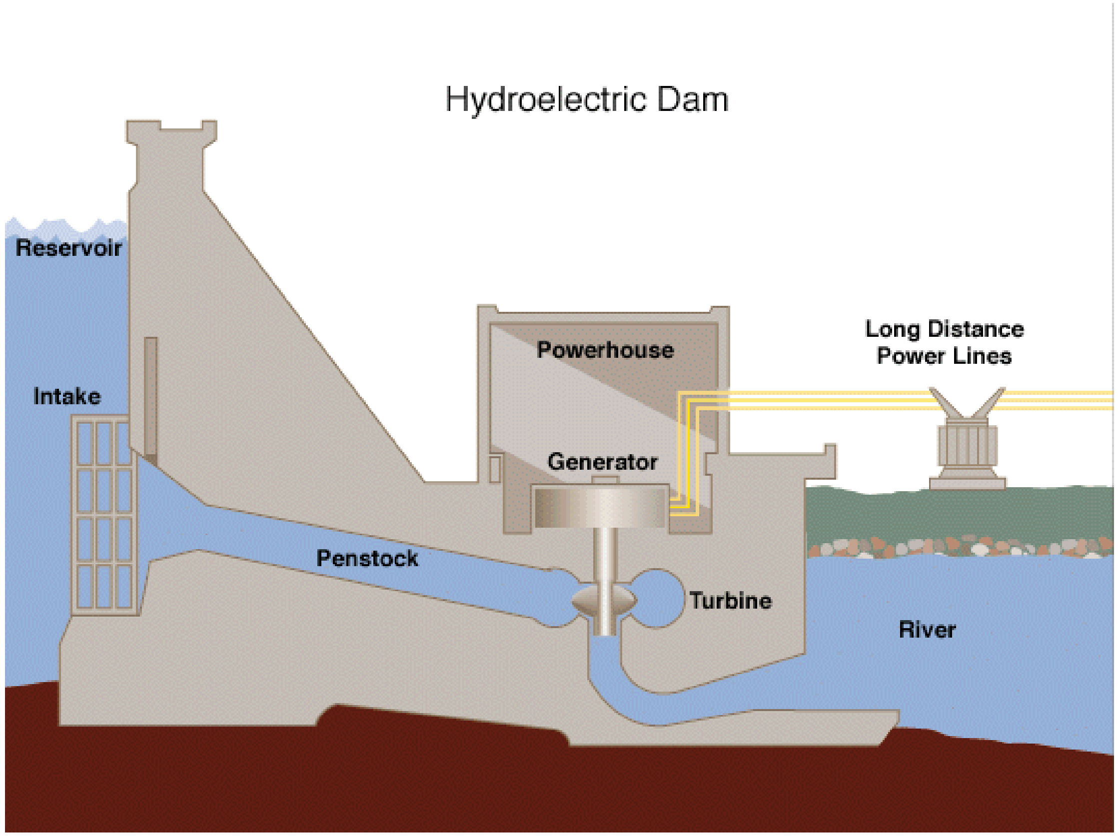


Research Topics: Biomedical Systems

Bloodflow: Diagnose and treat circulatory ailments; e.g., clots



Research Topics: Improved Hydroelectric Designs



Research Topics: Traffic Modeling and Control

Newton's Law of Viscosity

Deformation Rate: Rate of change in distance between two neighboring points moving with fluid divided by the distance between the points. That is, "change in length per unit length per unit time."

Strain: In solids, strain is the change in length per unit length.

Note: Deformation rate thus often referred to as strain rate.

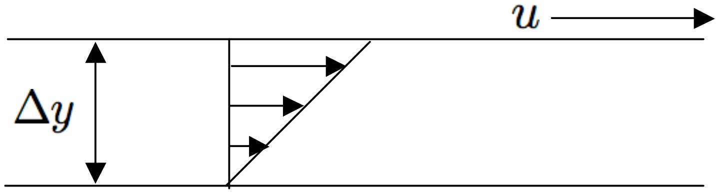
Shear Stress: Stress applied parallel or tangential to face of material; e.g., slide deck of cards.

Viscosity: Measure of the resistance of a fluid that is being deformed by a shear or tensional stress. The study of viscosity is termed rheology.

- High viscosity: honey, magma
- Low viscosity: air, superfluids
- Units: Ns/m^2 or 1 Poise (P) = 1 dyne s $/cm^2$ = 1 g/(cm s)
- E.g., Lubricating oils (~100 centipoise), water (~1 centipoise), air (~10e3-centipoise)

Newton’s Law of Viscosity

Newton’s Law of Viscosity: Consider a moving plate separated from fixed plate by fluid. For a “Newtonian fluid”, the force required to move the plate is proportional to the velocity and area and inversely proportional to distance between the plates.



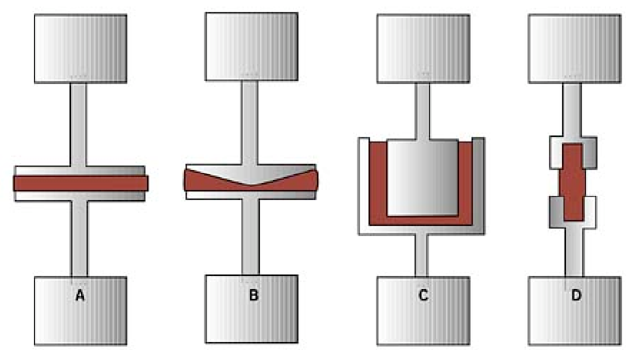
Relation:

$$F = \mu A \frac{\Delta u}{\Delta y}$$

$$\Rightarrow \tau_{yx} = \mu \frac{du}{dy}$$

Note: τ_{yx} is the stress acting the y-surface in the x-direction

Experiments: Couette Viscometer



Non-Newtonian Fluids

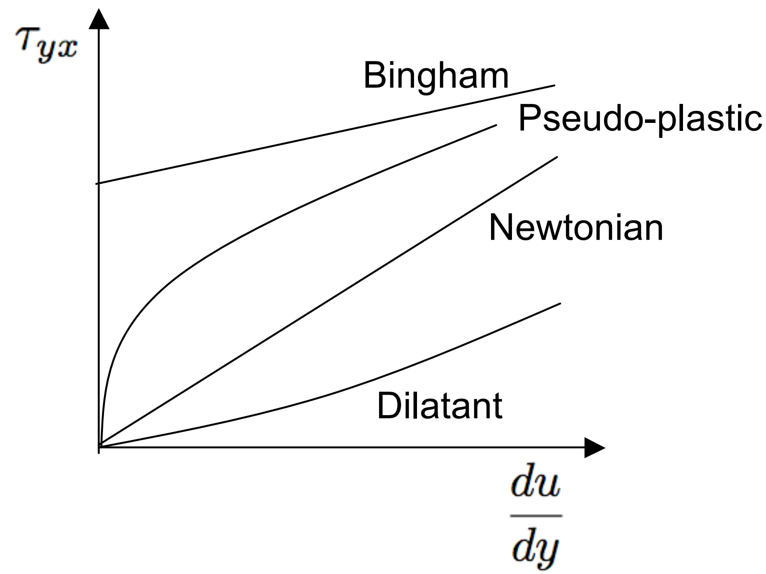
Newtonian Fluids: e.g., water, oils, glycerin, air and other gases at low to moderate shear rates

Bingham Fluids: exhibit yield stress; e.g., paint, ketchup

Pseudo-Plastic Fluids: shear-thinning fluids, often solutions of large, polymeric molecules in solution with smaller molecules; e.g., styling gel

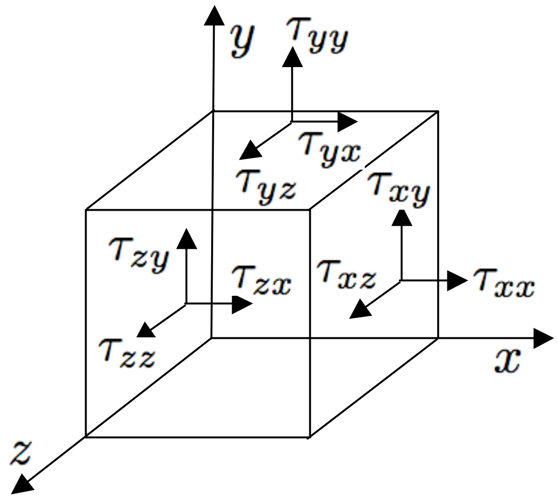
Dilatant Fluids: shear-thickening fluids which have increased viscosity at higher rates; e.g., uncooked paste of cornstarch and water, coupling fluids used in 2-wheel/4-wheel drive vehicles

Time-Dependent Viscosity: viscosity decreases with duration of stress; e.g., honey under certain conditions



Model Development: Shear Stresses

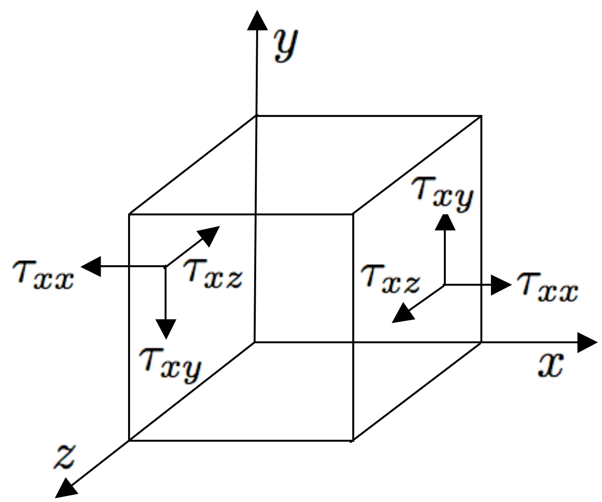
Stress Orientation:



Stress Tensor:

Sign Convention: Stress acting on coordinate plane having a positive outward normal normal is positive if the stress itself also acts in the positive direction. It is also considered positive if it acts in a negative direction on a surface with negative outward normal.

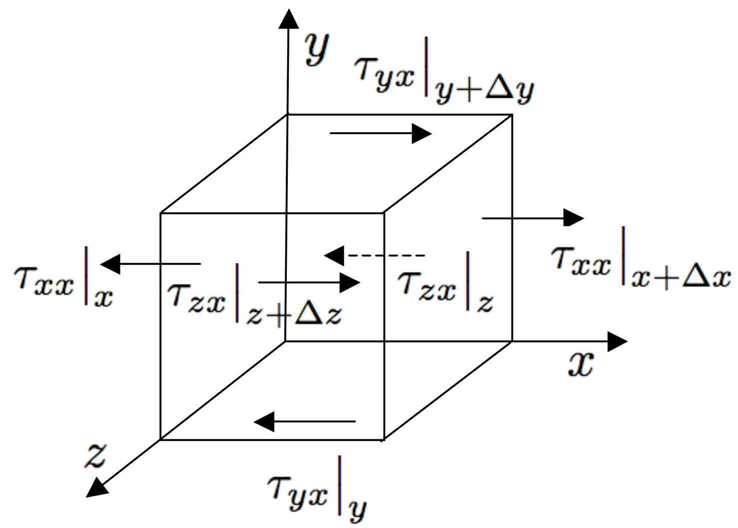
Note: Fluid at rest experiences only pressure which is a normal force that acts opposite to the outward normal.



Conservation of Momentum

Momentum Balance: x-direction

$$
\begin{aligned}
\frac{\partial(\rho u)}{\partial t}\Delta x \Delta y \Delta z \quad = \quad & \left[\rho u^2\big|_x - \rho u^2\big|_{x+\Delta x}\right]\Delta y \Delta z \\
+ \quad & \left[\rho v u\big|_y - \rho v u\big|_{y+\Delta y}\right]\Delta x \Delta z \\
+ \quad & \left[\rho w u\big|_z - \rho w u\big|_{z+\Delta z}\right]\Delta x \Delta y \\
+ \quad & \left[\tau_{xx}\big|_{x+\Delta x} - \tau_{xx}\big|_x\right]\Delta y \Delta z \\
+ \quad & \left[\tau_{yx}\big|_{y+\Delta y} - \tau_{yx}\big|_y\right]\Delta x \Delta z \\
+ \quad & \left[\tau_{zx}\big|_{z+\Delta z} - \tau_{zx}\big|_z\right]\Delta x \Delta y \\
+ \quad & \Delta y \Delta z \left[p\big|_x - p\big|_{x+\Delta x}\right]
\end{aligned}
$$



Momentum: x-component

$$
\frac{\partial(\rho u)}{\partial t} = -\left[\frac{\partial}{\partial x}(\rho u^2) + \frac{\partial}{\partial y}(\rho u v) + \frac{\partial}{\partial z}(\rho u w)\right] + \frac{\partial \tau_{xx}}{\partial x} + \frac{\partial \tau_{yx}}{\partial y} + \frac{\partial \tau_{zx}}{\partial z} - \frac{\partial p}{\partial x}
$$

Conservation of Momentum

Momentum: y-component

$$\frac{\partial(\rho v)}{\partial t} = -\left[\frac{\partial}{\partial x}(\rho u v) + \frac{\partial}{\partial y}(\rho v^2) + \frac{\partial}{\partial z}(\rho v w)\right] + \frac{\partial \tau_{xy}}{\partial x} + \frac{\partial \tau_{yy}}{\partial y} + \frac{\partial \tau_{zy}}{\partial z} - \frac{\partial p}{\partial y}$$

Momentum: z-component

$$\frac{\partial(\rho w)}{\partial t} = -\left[\frac{\partial}{\partial x}(\rho u w) + \frac{\partial}{\partial y}(\rho v w) + \frac{\partial}{\partial z}(\rho w^2)\right] + \frac{\partial \tau_{xz}}{\partial x} + \frac{\partial \tau_{yz}}{\partial y} + \frac{\partial \tau_{zz}}{\partial z} - \frac{\partial p}{\partial z}$$

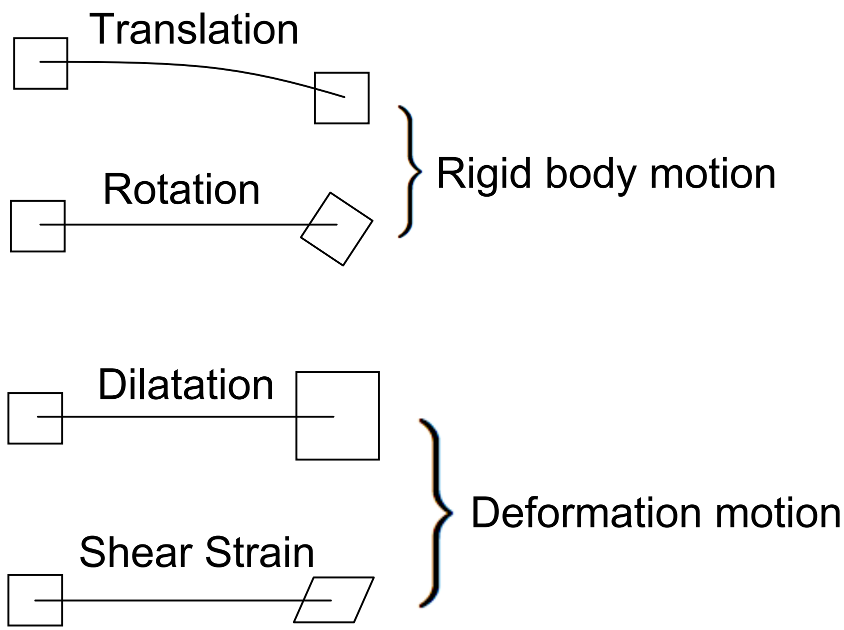
Note: Combination with the continuity equation yields

$$\rho \frac{D\vec{u}}{Dt} = -\nabla p + \nabla \cdot \tau$$

Because stress is a tensor, this is not a simple divergence.

Strain Rate Behavior

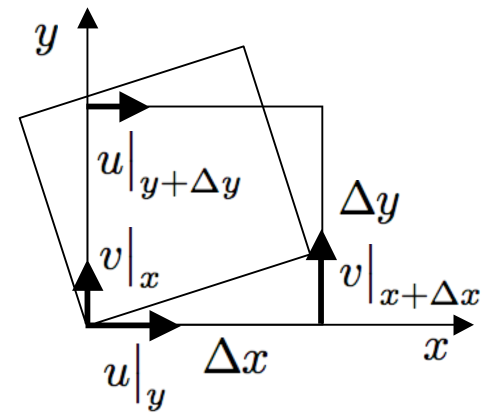
Goal: Generalize Newton's law of viscosity for 3-D flows. As a prelude, we first discuss the possible kinematics of a fluid element.



Viscous Effects: Angular deformation and dilatation

Strain Rate Behavior

Translation: Given by $\vec{u} = [u, v, w]$



Rotation:

- Angular velocity on side Δx:

$$\frac{v\big|_{x+\Delta x} - v\big|_{x}}{\Delta x}$$

- Angular velocity on side Δy:

$$-\frac{u\big|_{y+\Delta y} - u\big|_{y}}{\Delta y}$$

- Average and take limit to obtain angular velocity:

$$\Omega_z = \frac{1}{2}\left(\frac{\partial v}{\partial x} - \frac{\partial u}{\partial y}\right)$$

Similarly:

$$\Omega_x = \frac{1}{2}\left(\frac{\partial w}{\partial y} - \frac{\partial v}{\partial z}\right)$$

$$\Omega_y = \frac{1}{2}\left(\frac{\partial u}{\partial z} - \frac{\partial w}{\partial x}\right)$$

Relation to Vorticity:

$$\Omega = \frac{1}{2}(\nabla \times \vec{u}) = \frac{1}{2}\omega$$

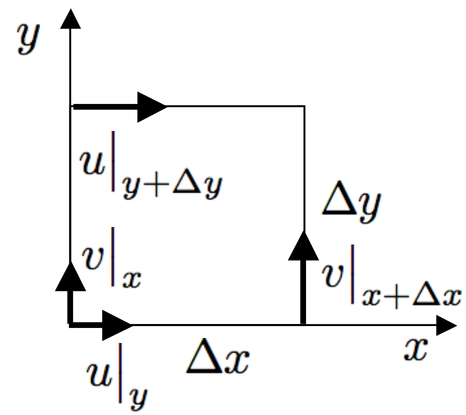
Strain Rate Behavior

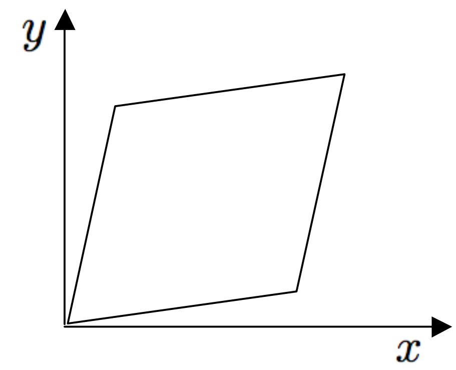
Shear Strain:

$$\dot{\epsilon}_{xy} = \frac{1}{2}\left(\frac{\partial u}{\partial y} + \frac{\partial v}{\partial x}\right)$$

$$\dot{\epsilon}_{yz} = \frac{1}{2}\left(\frac{\partial v}{\partial z} + \frac{\partial w}{\partial y}\right)$$

$$\dot{\epsilon}_{xz} = \frac{1}{2}\left(\frac{\partial w}{\partial x} + \frac{\partial u}{\partial z}\right)$$

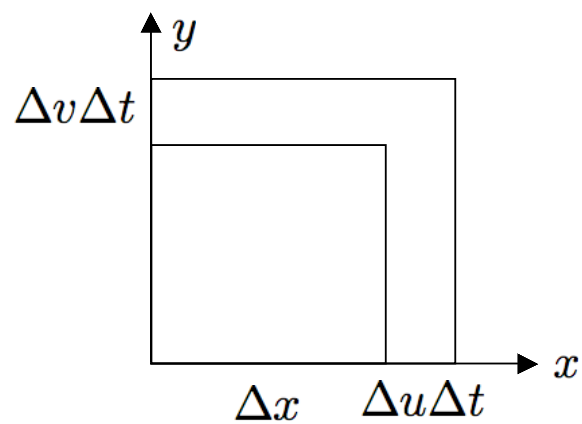




Dilatation: Increased volume

$$\Delta u \Delta t \Delta y \Delta z + \Delta v \Delta t \Delta x \Delta z + \Delta w \Delta t \Delta x \Delta y + \mathcal{O}(\Delta t^2)$$

$$\Rightarrow \phi = \frac{\partial u}{\partial x} + \frac{\partial v}{\partial y} + \frac{\partial w}{\partial z} = \nabla \cdot \vec{u}$$



Incompressible Fluid: $\phi = 0$

- e.g., water, air at supersonic speeds

Strain Rate Relations

Velocity Gradient Tensor:

$$\frac{\partial u_i}{\partial x_j} = \dot{e}_{ij} + \Omega_{ij}$$

Here

$$\dot{e}_{xx} = \frac{\partial u}{\partial x}\,; \qquad \dot{e}_{xy} = \dot{e}_{yx} = \frac{1}{2}\left(\frac{\partial u}{\partial y} + \frac{\partial v}{\partial x}\right)$$

$$\dot{e}_{yy} = \frac{\partial v}{\partial y}\,; \qquad \dot{e}_{yz} = \dot{e}_{zy} = \frac{1}{2}\left(\frac{\partial v}{\partial z} + \frac{\partial w}{\partial y}\right)$$

$$\dot{e}_{zz} = \frac{\partial w}{\partial z}\,; \qquad \dot{e}_{xz} = \dot{e}_{zx} = \frac{1}{2}\left(\frac{\partial w}{\partial x} + \frac{\partial u}{\partial z}\right)$$

$$\Omega_x = \Omega_{zy} = -\Omega_{yz} = \frac{1}{2}\left(\frac{\partial w}{\partial y} - \frac{\partial v}{\partial z}\right)$$

$$\Omega_y = \Omega_{xz} = -\Omega_{zx} = \frac{1}{2}\left(\frac{\partial u}{\partial z} - \frac{\partial w}{\partial x}\right)$$

$$\Omega_z = \Omega_{yx} = -\Omega_{xy} = \frac{1}{2}\left(\frac{\partial v}{\partial x} - \frac{\partial u}{\partial y}\right)$$

Constitutive Relations

Stress-Strain Rate Relations:

$$\tau_{ij} = 2\mu\dot{e}_{ij} + \delta_{ij}\lambda\phi \qquad \mu, \lambda\text{: Viscosity coefficients}$$

Define $\zeta = \lambda + \frac{2}{3}\mu$ to obtain

$$\begin{aligned}\tau_{ij} &= \mu\left(\frac{\partial u_i}{\partial x_j} + \frac{\partial u_j}{\partial x_i} - \frac{2}{3}\delta_{ij}\frac{\partial u_\ell}{\partial x_\ell}\right) + \zeta\delta_{ij}\frac{\partial u_\ell}{\partial x_\ell} \\ &= \mu\left(\frac{\partial u_i}{\partial x_j} + \frac{\partial u_j}{\partial x_i}\right) - \left(\frac{2}{3}\mu - \zeta\right)\delta_{ij}\frac{\partial u_\ell}{\partial x_\ell}\end{aligned}$$

Note: In component form, this yields

$$\tau_{xx} = 2\mu\frac{\partial u}{\partial x} - \left(\frac{2}{3}\mu - \zeta\right)\nabla\cdot\vec{u} \qquad \tau_{xy} = \tau_{yx} = \mu\left(\frac{\partial u}{\partial y} + \frac{\partial v}{\partial x}\right)$$

$$\tau_{yy} = 2\mu\frac{\partial v}{\partial y} - \left(\frac{2}{3}\mu - \zeta\right)\nabla\cdot\vec{u} \qquad \tau_{yz} = \tau_{zy} = \mu\left(\frac{\partial v}{\partial z} + \frac{\partial w}{\partial y}\right)$$

$$\tau_{zz} = 2\mu\frac{\partial w}{\partial z} - \left(\frac{2}{3}\mu - \zeta\right)\nabla\cdot\vec{u} \qquad \tau_{zx} = \tau_{xz} = \mu\left(\frac{\partial w}{\partial x} + \frac{\partial u}{\partial z}\right)$$

Navier Stokes Equations

Strategy: Employ stress relations in momentum equations

$$\rho\left(\frac{\partial u_i}{\partial t}+u_j\frac{\partial u_i}{\partial x_j}\right)=-\frac{\partial p}{\partial x_i}+\frac{\partial}{\partial x_j}\left[\mu\left(\frac{\partial u_i}{\partial x_j}+\frac{\partial u_j}{\partial x_i}-\frac{2}{3}\delta_{ij}\frac{\partial u_\ell}{\partial x_\ell}\right)\right]+\frac{\partial}{\partial x_i}\left(\zeta\frac{\partial u_\ell}{\partial x_\ell}\right)$$

E.g., First component

$$\begin{aligned}\rho\left(\frac{\partial u}{\partial t}+u\frac{\partial u}{\partial x}+v\frac{\partial v}{\partial y}+w\frac{\partial w}{\partial z}\right) &= -\frac{\partial p}{\partial x}+\frac{\partial}{\partial x}\left[2\mu\frac{\partial u}{\partial x}-\left(\frac{2}{3}\mu-\zeta\right)\nabla\cdot\vec{u}\right] \\ &\quad +\frac{\partial}{\partial y}\left[\mu\left(\frac{\partial u}{\partial y}+\frac{\partial v}{\partial x}\right)\right]+\frac{\partial}{\partial z}\left[\mu\left(\frac{\partial w}{\partial x}+\frac{\partial u}{\partial z}\right)\right]\end{aligned}$$

Note: These equations are often simplified through various assumptions

- e.g., Bulk viscosity λ typically neglected in dense gasses and liquids
- Incompressible fluids: $\phi=\nabla\cdot\vec{u}=0$. If constant μ,

$$\rho\frac{D\vec{u}}{Dt}=-\nabla p+\mu\nabla^2\vec{u}$$

Euler and Burgers' Equations

Euler Equations: Used when viscosity is negligible (inviscid fluids)

$$\rho \frac{D\vec{u}}{Dt} = -\nabla p$$

Inviscid Burgers' Equation: Nonlinear equation illustrates formation of shocks

$$\frac{\partial u}{\partial t} + u\frac{\partial u}{\partial x} = 0$$ Quasilinear Form

$$\frac{\partial u}{\partial t} + \frac{\partial}{\partial x}\left(\frac{1}{2}u^2\right) = 0$$ Conservation Form

Burgers' Equation: Combines nonlinear behavior and dissipation. Provides a simplified model for analysis and testing numerical methods

$$\frac{\partial u}{\partial t} + u\frac{\partial u}{\partial x} = \mu\frac{\partial^2 u}{\partial x^2}$$

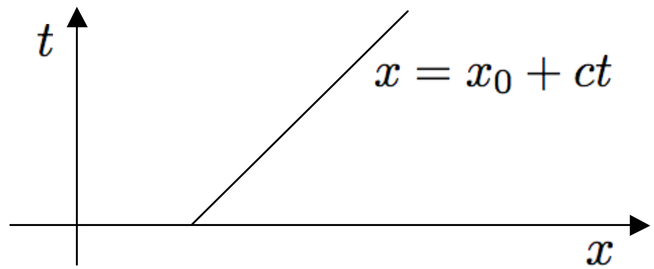
Advection Equation

Advection Equation: Constant wave speed c

$$u_t + cu_x = 0$$

$$u(0, x) = \phi(x)$$

$$\Rightarrow u(t, x) = \phi(x - ct)$$



- Solution constant along characteristics: $x - ct = x_0$

Characteristics: Satisfy

$$x'(t) = c$$

$$x(0) = x_0$$

Differentiation along characteristics yields

$$\begin{aligned} \frac{d}{dt} u(t, x(t)) &= \frac{\partial u}{\partial t} + \frac{\partial u}{\partial x} \frac{dx}{dt} \\ &= u_t + cu_x \\ &= 0 \end{aligned}$$

Conclusion: $u(t, x)$ is constant along characteristic in this case

Advection Equation

Advection Equation: Variable wave speed c(x)

Characteristic Equations:

$$x'(t) = c(x(t))$$

$$x(0) = x_0$$

Nonconstant Solution: Solution to differential equation

$$\frac{d}{dt}u(t, x(t)) = -c'(x)u(t, x(t))$$

$$u(0, x) = \phi(x)$$

Inviscid Burgers Equation

Burgers Equation:

$$u_t + uu_x = 0$$

$$u(0,x) = \phi(x)$$

Characteristic Equations:

$$x'(t) = u(t, x(t))$$

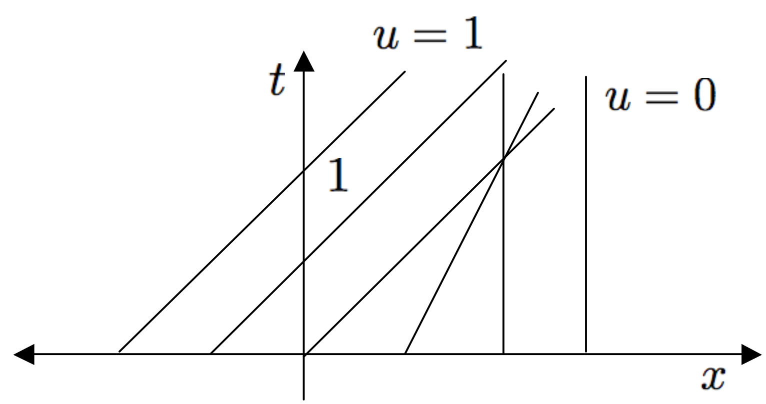
$$x(0) = x_0$$

$$\Rightarrow x(t) = \xi + u(0,\xi)t$$

Solutions:

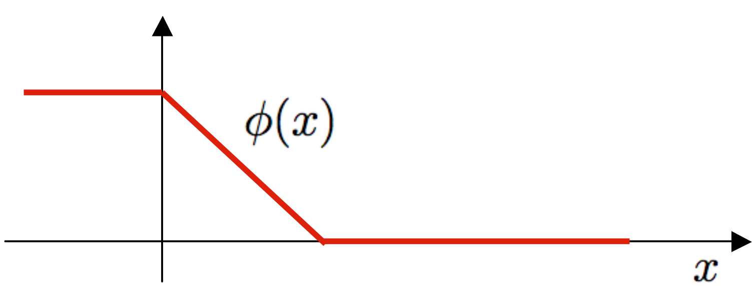
$$u'(t) = u_t + uu_x = 0$$

$$\Rightarrow u(t,x) = u(0,\xi)$$



Example:

$$\phi(x) = \begin{cases} 1\,, & x \leq 0 \\ 1 - x\,, & 0 < x < 1 \\ 0\,, & x \geq 0 \end{cases}$$



Question: How do we uniquely specify solution after t=1?

Rankine-Hugoniot Jump Condition

Conservation Relation: (Differential form)

$$\frac{\partial u}{\partial t} + \frac{\partial f}{\partial x} = 0$$

Conservation Relation: (Integral form)

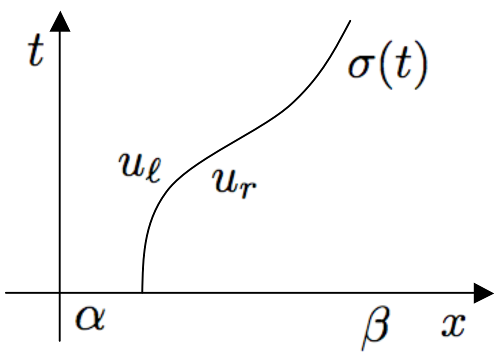
$$\frac{d}{dt}\int_V u dx = -\int_S f \cdot \hat{n} ds$$

$$\Rightarrow \frac{d}{dt}\int_\alpha^\beta u dx = -f(u(t,\beta)) + f(u(t,\alpha)) \quad (\dagger)$$

Goal: Determine a curve $x = \sigma(t)$ that characterizes the evolution of the discontinuity. Take

$$\lim_{x \to \sigma(t)^-} u(t,x) = u_\ell$$

$$\lim_{x \to \sigma(t)^+} u(t,x) = u_r$$



Jump: $[u] \equiv u_\ell - u_r$

Note: Choose interval $[\alpha, \beta]$ which contains $\sigma(t)$ at time t

Rankine-Hugoniot Jump Condition

Note:

$$\begin{aligned}
\text{LHS of } (\dagger) &= \frac{d}{dt}\left[\int_{\alpha}^{\sigma(t)^-} u(t,x)dx + \int_{\sigma(t)+}^{\beta} u(t,x)dx\right] \\
&= \int_{\alpha}^{\sigma(t)^-} u_t(t,x)dx + u_\ell\dot{\sigma} + \int_{\sigma(t)+}^{\beta} u(t,x)dx - u_r\dot{\sigma} \quad \text{(Leibniz Rule)} \\
&= [u]\dot{\sigma} - f(u)\big|_{\alpha}^{\sigma(t)^-} - f(u)\big|_{\sigma(t)+}^{\beta}
\end{aligned}$$

From (†), this yields

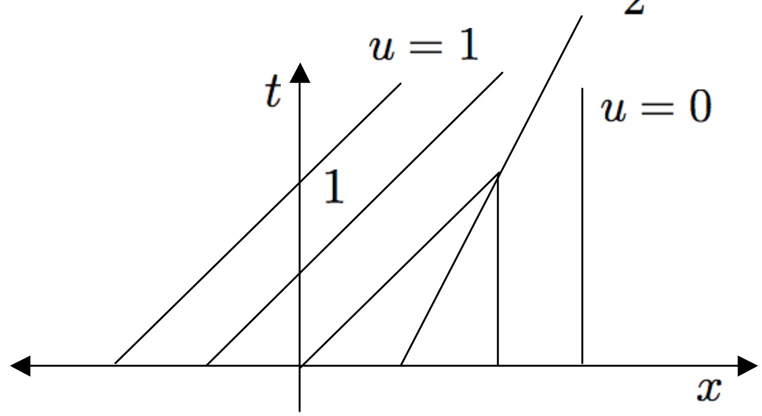
$$[u]\dot{\sigma} = [f]$$

Shock Speed:

$$s = \frac{f(u_\ell) - f(u_r)}{u_\ell - u_r}$$

Previous Example: $f(u) = \frac{1}{2}u^2 \Rightarrow \dot{\sigma} = \frac{1}{2}$

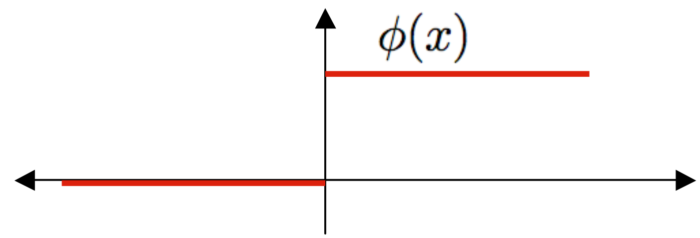
$$\sigma(t) = \frac{1}{2}(t+1)$$

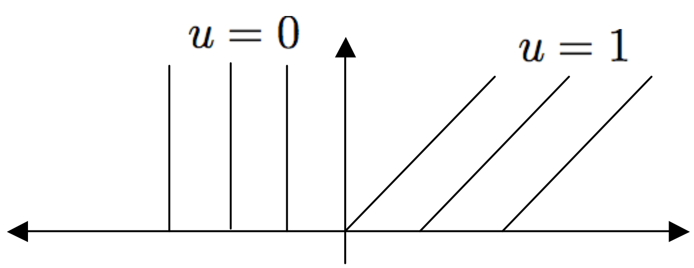


Entropy Condition

Example: Consider the inviscid Burgers equation with initial data

$$\phi(x) = \begin{cases} 0 \,, x < 0 \\ 1 \,, x > 0 \end{cases}$$





Note: No characteristics in the region $\{0 < x < t, t > 0\}$

Entropy Condition: Require

$$a(u_r) < \dot{\sigma} < a(u_\ell)$$

$$\Leftrightarrow f'(u_r) < s < f'(u_\ell)$$

Definition: A curve that satisfies

$$[u]\dot{\sigma} = [f]$$

$$a(u_r) < \dot{\sigma} < a(u_\ell)$$

is termed a shock. Resulting solutions u are termed generalized solutions.

Traffic Flow Example

Model:

$$\rho_t + (\rho u)_x = 0$$

where ρ has units of cars/mile. To relate u and ρ, consider the expression

$$u(\rho) = u_{\max}\left(1 - \rho/\rho_{\max}\right)$$

$$\Rightarrow f(\rho) = \rho u_{\max}\left(1 - \rho/\rho_{\max}\right)$$

Characteristic Speed:

$$f'(\rho) = u_{\max}\left(1 - 2\rho/\rho_{\max}\right)$$

Shock Speed:

$$\dot{\sigma} = \frac{f(\rho_\ell) - f(\rho_r)}{\rho_\ell - \rho_r} = u_{\max}\left[1 - \frac{(\rho_\ell + \rho_r)}{\rho_{\max}}\right]$$

Entropy Condition:

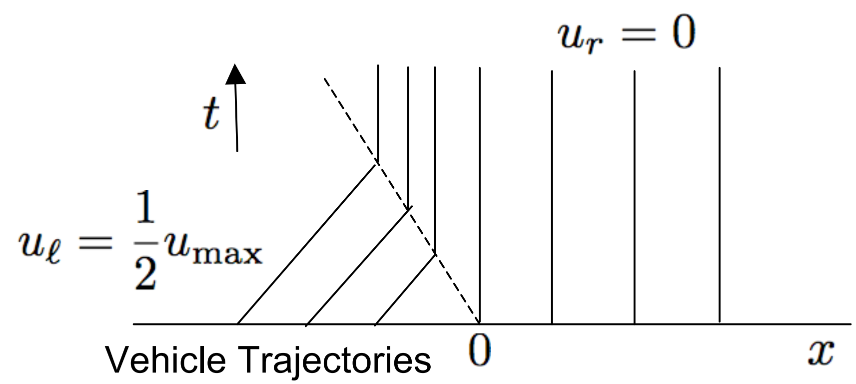
$$f'(\rho_\ell) > f'(\rho_r) \Rightarrow \rho_\ell < \rho_r$$

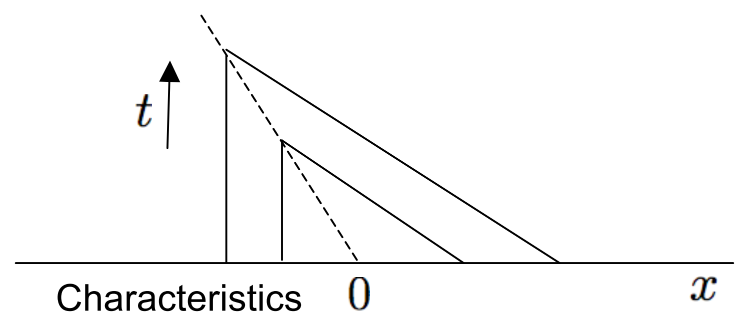
Traffic Flow Example

Case 1: Initial density

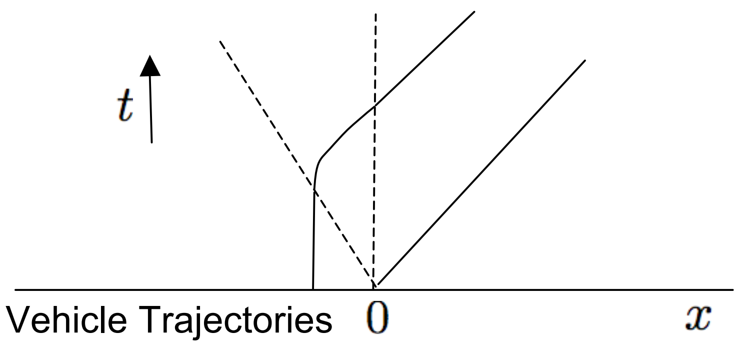
$$\rho(t,0) = \begin{cases} \rho_l \ , x < 0 \\ \rho_r \ , x > 0 \end{cases}$$

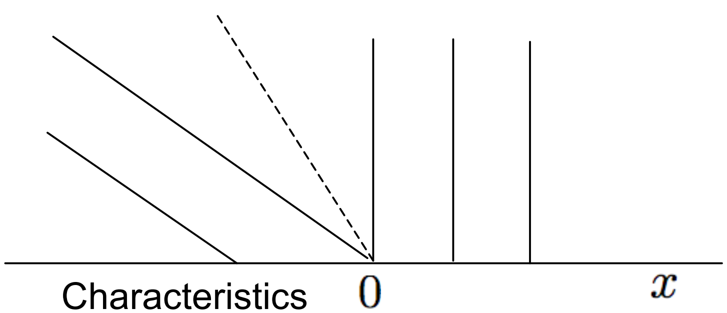
where $\rho_\ell = \frac{1}{2}\rho_{\max}$ and $\rho_r = \rho_{\max}$.





Case 2: $\rho_\ell = \rho_{\max}, \rho_r = \frac{1}{2}\rho_{\max}$





Numerical Methods for Burgers Equation

Conservation Form:

$$\frac{\partial u}{\partial t} + \frac{\partial}{\partial x}\left(\frac{1}{2}u^2\right) = 0$$

First-Order Differencing:

$$u_{i,j+1} = u_{i,j} - \frac{k}{h}\left[\frac{1}{2}u_{i,j}^2 - \frac{1}{2}u_{i-1,j}^2\right]$$

Note: You will often get an incorrect shock speed with the quasilinear approximation

$$u_{i,j+1} = u_{i,j} - \frac{k}{h}u_{i,j}\left[u_{i,j} - u_{i-1,j}\right]$$

General Conservation Relation: $u_t + (f(u))_x = 0$

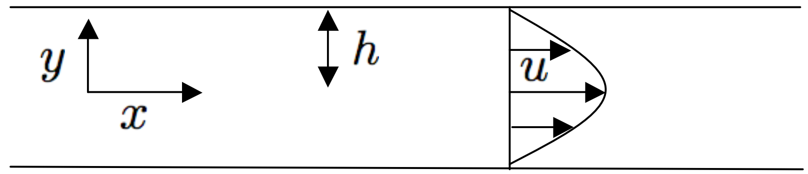
$$u_{i,j+1} = u_{i,j} - \frac{k}{h}\left[f(u_{i,j}) - f(u_{i-1,j})\right] \ , \ f'(u) > 0$$

$$u_{i,j+1} = u_{i,j} - \frac{k}{h}\left[f(u_{i+1,j}) - f(u_{i,j})\right] \ , \ f'(u) < 0$$

Note: More comprehensive methods need to be employed if f'(u) switches sign.

Poiseuille Flows

Parallel Plates:



Note: Velocity a function only of y implies $\frac{\partial u}{\partial y} = \frac{du}{dy}$. Thus

$$0 = -\frac{dp}{dx} + \mu \frac{d^2u}{dy^2}$$

$$\Rightarrow \mu \frac{du}{dy} = \frac{dp}{dx} + c_1$$

Note: $\left.\frac{du}{dy}\right|_{y=0} = 0 \Rightarrow c_1 = 0$

$$\Rightarrow \tau = \tau_{yx} = \mu \frac{du}{dy} = \frac{dp}{dx}$$

Note: Shear stress is a linear function of y

Moreover,

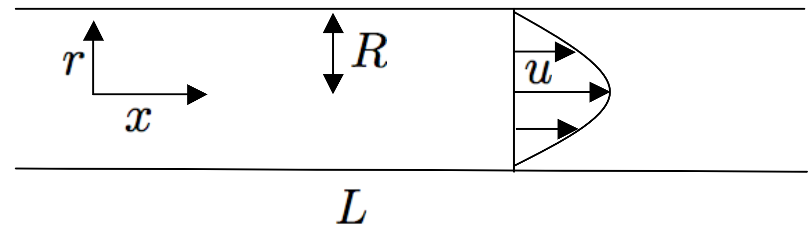
$$u = \frac{1}{2\mu}\frac{dp}{dx}y^2 + c_2$$

$$\Rightarrow u(y) = \frac{1}{2\mu}\frac{dp}{dx}(y^2 - h^2) \quad \text{since} \quad u\big|_{y=\pm h} = 0$$

Hagen-Poiseuille Flows

Pipe Flow: Analogous development

$$u(r) = \frac{1}{4\mu}\frac{dp}{dx}(r^2 - R^2)$$



Notation:

$\mathcal{U}$: Volumetric flow rate

$\mathcal{U} = \pi R^2 \bar{u}$

Note:

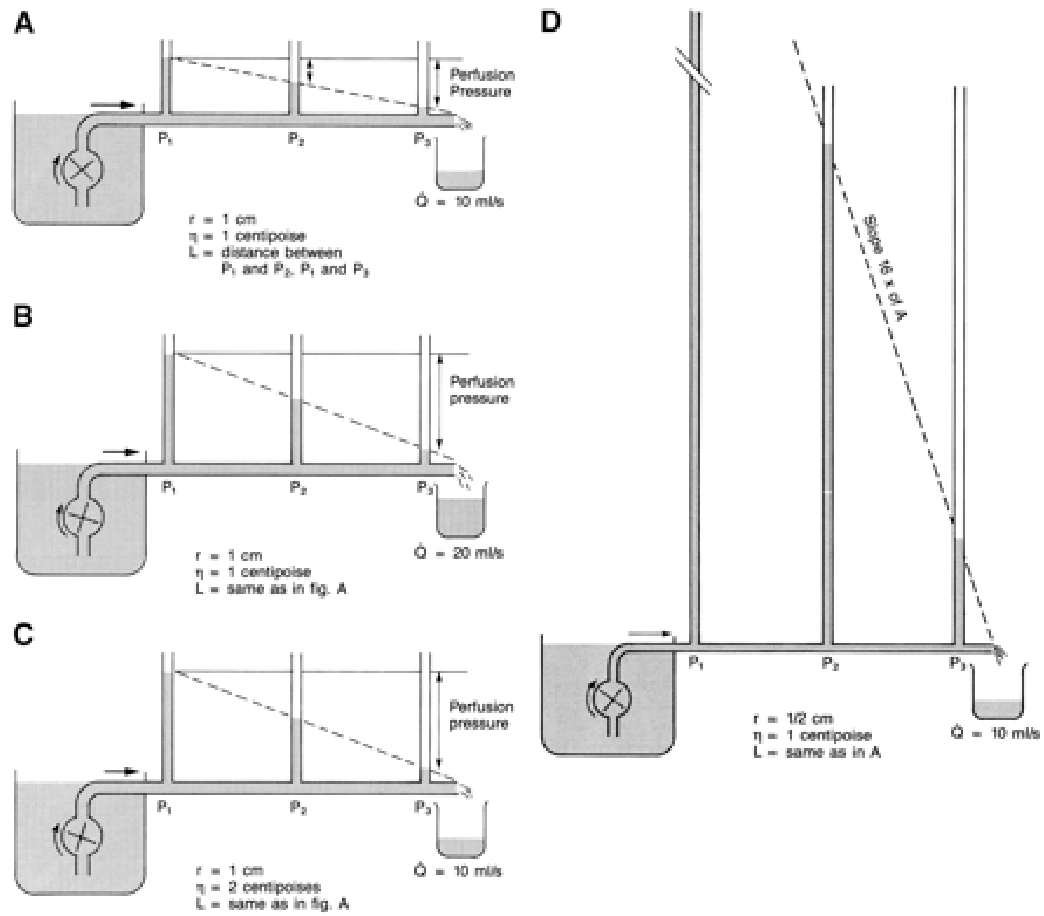
$$\mathcal{U} = \int_0^R \frac{1}{4\mu}\frac{dp}{dx}(r^2 - R^2)2\pi r dr$$

$$= \frac{\pi R^4}{8\mu}\frac{dp}{dx}$$

Hagen-Poiseuille Equation:

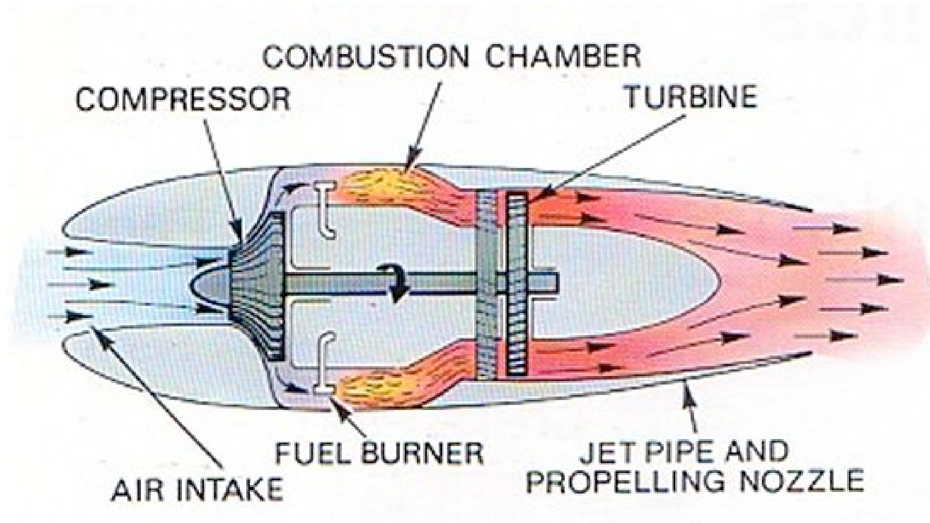
$$\Delta P = \frac{8\mu L \mathcal{U}}{\pi R^4}$$

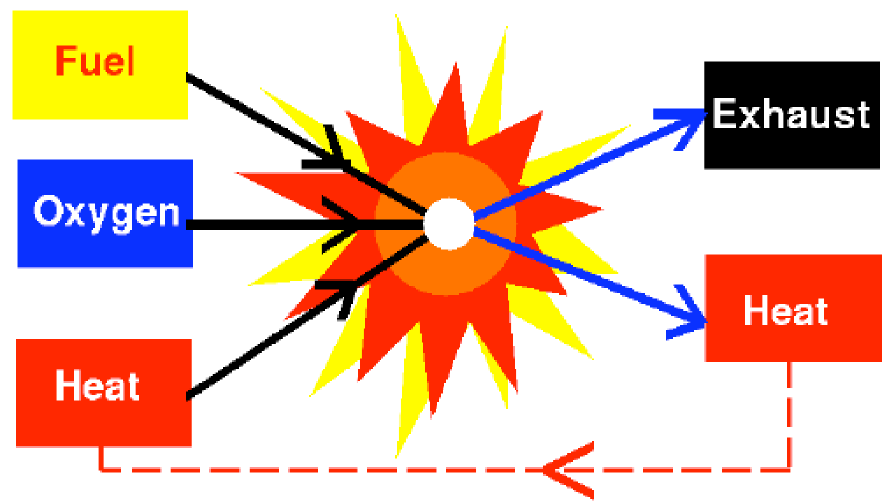
$$\mathcal{R} = \frac{8\mu L}{\pi R^4}$$ Resistance to flow between two points



Conservation of Energy

Note: Must include energy balance if heat or work affects the flow.





Conservation of Energy

Definition: The energy of a body is defined as the capacity of the body to do work. The units of energy are the same as those of work.

Definition: The kinetic energy is that due to its motion.

Definition: The internal energy is that stored within the substance and is due to the activity and spacing of molecules.

Definition: The heat or heat transfer is defined as thermal energy in transition due to a temperature difference between two points.

Principle:

$$\left\{\begin{array}{l}\text{Increase in energy (internal}\\ \text{and kinetic) of unit mass}\end{array}\right\} = \left\{\begin{array}{l}\text{Rate at which heat is}\\ \text{transferred to the body}\end{array}\right\} + \left\{\begin{array}{l}\text{Rate which surface work is}\\ \text{done on the body}\end{array}\right\} + \left\{\begin{array}{l}\text{Rate at which body forces}\\ \text{(e.g., gravity, EM) do work}\\ \text{on the body}\end{array}\right\}$$

Integrate in Time:

$$\Delta E = \Delta(U + KE + PE) = Q - W$$

First Law of Thermodynamics

Conservation of Energy

Notation:

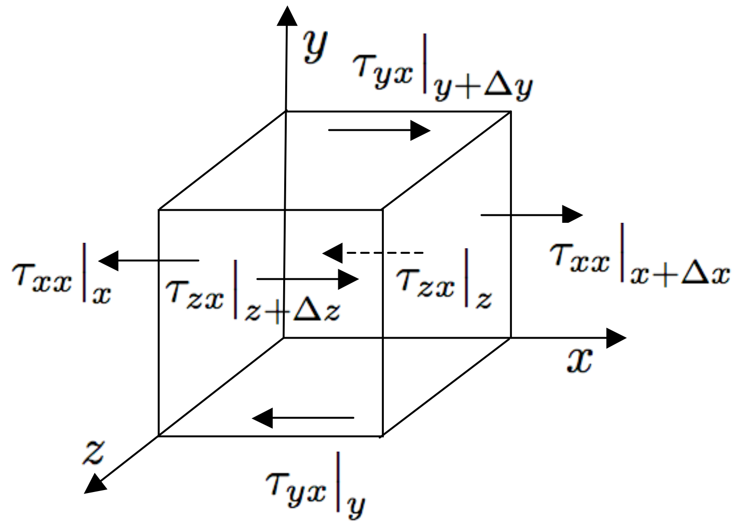
e: Internal energy/unit mass – Units: J/kg

q: Heat flux – Units: W/m^2 = J/(s m^2)

$$\Rightarrow \dot{Q} = -\int_S q \cdot \hat{n} dS$$

ρ: Fluid density – Units: kg/m^3

$\vec{u} = [u, v, w]$: Velocity field



Energy Balance: Flow in x-direction

$$\begin{aligned}
\frac{D}{Dt}\int_V \rho(e + \frac{1}{2}u^2 + gz)dV \;&=\; -\int_S q \cdot \hat{n} dS + \Delta y \Delta z \left[pu\big|_x - pu\big|_{x+\Delta x}\right] \\
&+\; \left[\tau_{xx}u\big|_{x+\Delta x} - \tau_{xx}u\big|_x\right] \Delta y \Delta z \\
&+\; \left[\tau_{yx}u\big|_{y+\Delta y} - \tau_{yx}u\big|_y\right] \Delta x \Delta z \\
&+\; \left[\tau_{zx}u\big|_{z+\Delta z} - \tau_{zx}u\big|_z\right] \Delta x \Delta y
\end{aligned}$$

Conservation of Energy

Note:

$$\frac{D}{Dt}\int_V \rho(e+\frac{1}{2}u^2+gz)dV = \int_V \left[\frac{\partial}{dt}(\rho\tilde{e})+u\frac{\partial}{\partial x}(\rho\tilde{e})\right]dV \quad , \quad \tilde{e}=e+\frac{1}{2}u^2+gz$$

$$= \int_V \left[\left(\frac{\partial \rho}{\partial t}+u\frac{\partial \rho}{\partial x}\right)\tilde{e}+\rho\left(\frac{\partial \tilde{e}}{\partial t}+u\frac{\partial \tilde{e}}{\partial x}\right)\right]dV$$

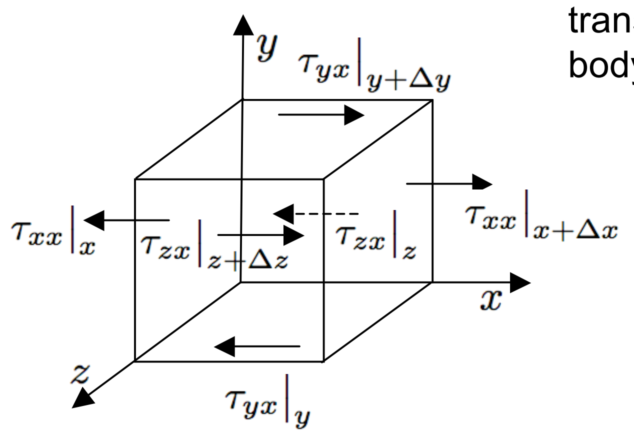
$$= \int_V \rho\frac{D\tilde{e}}{Dt}dV$$

Energy Relation: Flow in x-direction

$$\rho\frac{D}{Dt}\left(e+\frac{1}{2}u^2+gz\right) = -\nabla\cdot q+\frac{\partial}{\partial x}(u\tau_{xx})+\frac{\partial}{\partial y}(u\tau_{yx})+\frac{\partial}{\partial z}(u\tau_{zx})-\frac{\partial}{\partial x}(pu)$$

Heat transferred to body

Surface work done on body



Note: Can simplify using momentum equation

Conservation of Energy

Energy Relation: General flow $\vec{u} = [u, v, w]$

$$\frac{D}{Dt}\int_V \rho\left(e + \frac{1}{2}|\vec{u}|^2 + gz\right) dV = -\int_S q \cdot \hat{n} dS + \int_S \vec{t}_{(n)} \cdot \vec{u} dS$$

Here the dyad $\vec{t}_{(n)} = \hat{n} \cdot \mathcal{T}$ is defined as

$$\begin{aligned}\vec{t}_{(n)} = \;& \hat{\imath}\left[n_x \mathcal{T}_{xx} + n_y \mathcal{T}_{yx} + n_z \mathcal{T}_{zx}\right] \\ & +\hat{\jmath}\left[n_x \mathcal{T}_{xy} + n_y \mathcal{T}_{yy} + n_z \mathcal{T}_{zy}\right] \\ & +\hat{k}\left[n_x \mathcal{T}_{xz} + n_y \mathcal{T}_{yz} + n_z \mathcal{T}_{zz}\right]\end{aligned}$$

where

$$\hat{n} = \hat{\imath} n_x + \hat{\jmath} n_y + \hat{k} n_z$$

$$\mathcal{T} = \tau - pI$$

$$\tau = \begin{bmatrix} \tau_{xx} & \tau_{xy} & \tau_{xz} \\ \tau_{yx} & \tau_{yy} & \tau_{yz} \\ \tau_{zx} & \tau_{zy} & \tau_{zz} \end{bmatrix}$$

Conservation of Energy

Energy Relation: Differential form

$$\rho \frac{D}{Dt}\left(e + \frac{1}{2}u^2 + gz\right) = -\nabla \cdot q + \nabla \cdot (\mathcal{T} \cdot \vec{u})$$

Here

$$\begin{aligned} \mathcal{T} \cdot \vec{u} = \quad & \hat{\imath}\left[\mathcal{T}_{xx}u + \mathcal{T}_{xy}v + \mathcal{T}_{xz}w\right] \\ & +\hat{\jmath}\left[\mathcal{T}_{yx}u + \mathcal{T}_{yy}v + \mathcal{T}_{yz}w\right] \\ & +\hat{k}\left[\mathcal{T}_{zx}u + \mathcal{T}_{zy}v + \mathcal{T}_{zz}w\right] \end{aligned}$$

Temperature Relation

Note: $H = c_p m T$

$$\Rightarrow \rho e = \frac{H}{m^3} = c_p \rho T$$

Fourier's Law:

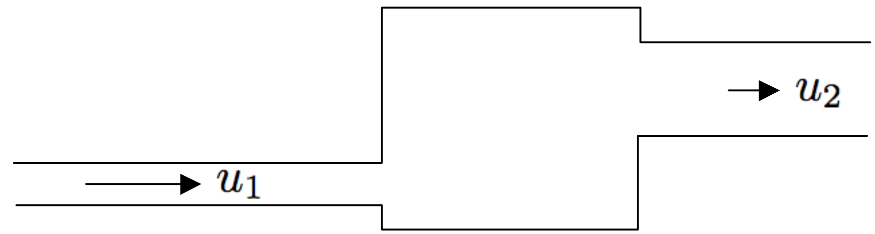
$$q = -k \nabla T \cdot \hat{n}$$

Assumptions: Negligible shear stresses

$$\rho c_p \frac{DT}{Dt} = k \nabla^2 T - \frac{\partial}{\partial x}(pu)$$

Bernoulli's Principle

Assumptions: Steady,1-D flow



Energy Relation:

$$\frac{\partial Q}{\partial t} - \frac{\partial W}{\partial t} = \int_{S_1+S_2} \left(\frac{p}{\rho} + e + \frac{1}{2}u^2 + gz \right) \rho u dS$$
$$= \left(\frac{p_2}{\rho_2} + e_2 + \frac{1}{2}u_2^2 + gz_2 \right) \rho_2 A_2 u_2 - \left(\frac{p_1}{\rho_1} + e_1 + \frac{1}{2}u_1^2 + gz_1 \right) \rho_1 A_1 u_1$$

Conservation of Mass:

$$\frac{dm}{dt} = \rho_1 A_1 u_1 = \rho_2 A_2 u_2$$

Combination yields

$$\frac{\partial Q}{\partial t} - \frac{\partial W}{\partial t} = \left[\left(\frac{p_2}{\rho_2} - \frac{p_1}{\rho_1} \right) + (e_2 - e_1) + \frac{u_2^2 - u_1^2}{2} g(z_2 - z_1) \right] \frac{dm}{dt}$$

Bernoulli's Principle

Reformulation in terms of change per unit mass yields

$$q - w = \left(\frac{p_2}{\rho_2} - \frac{p_1}{\rho_1}\right) + (e_2 - e_1) + \frac{u_2^2 - u_1^2}{2} + g(z_2 - z_1)$$

Assumption: Incompressible flow

$$\begin{aligned} -w &= \frac{p_2 - p_1}{\rho} + \frac{u_2^2 - u_1^2}{2} + g(z_2 - z_1) + (e_2 - e_1 - q) \\ &= \frac{p_2 - p_1}{\rho} + \frac{u_2^2 - u_1^2}{2} + g(z_2 - z_1) + gH_L \end{aligned}$$

where the "head loss"

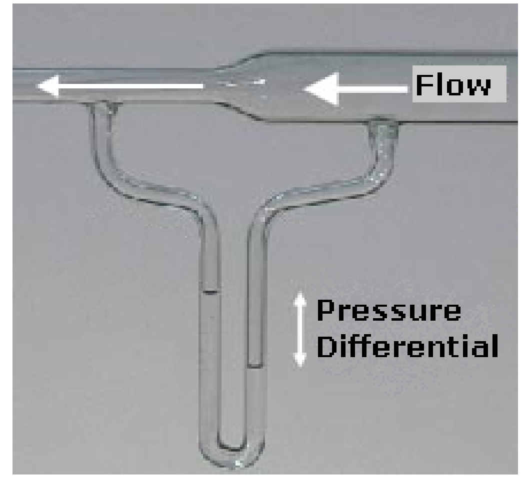
$$gH_L = e_2 - e_1 - q$$

represents conversion of mechanical to thermal energy.

Zero Work Case:

$$\frac{p_2 - p_1}{\rho} + \frac{u_2^2 - u_1^2}{2} + g(z_2 - z_1) = 0$$

Bernoulli's Principle



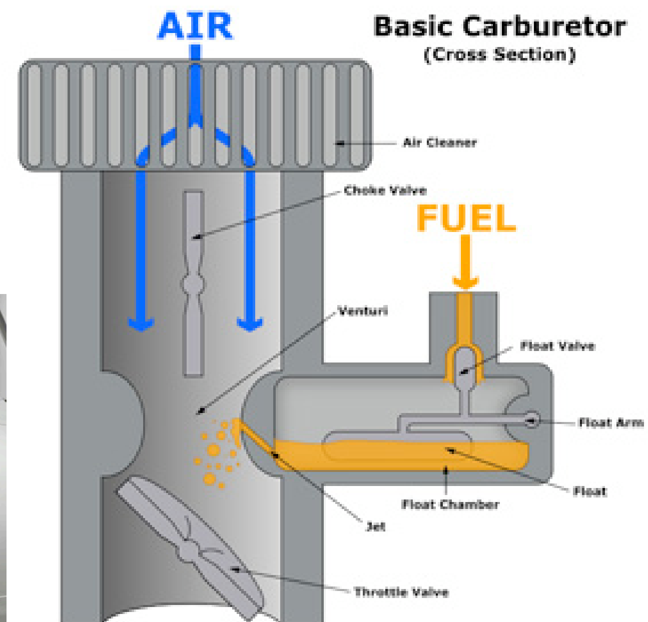
Venturi meter

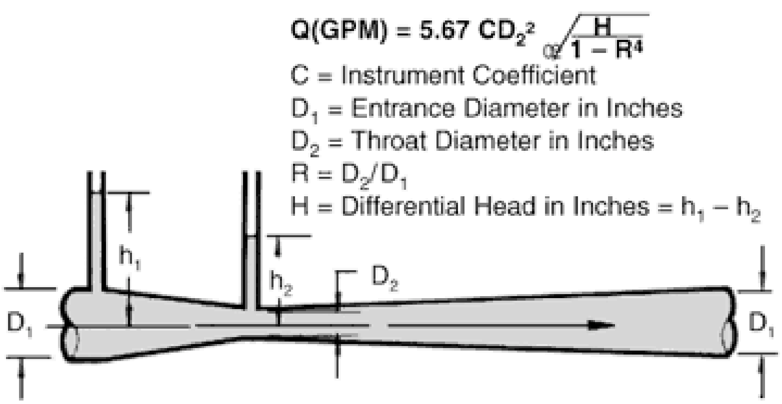
Common Form:

$$\frac{u^2}{2} + gz + \frac{p}{\rho} = C$$

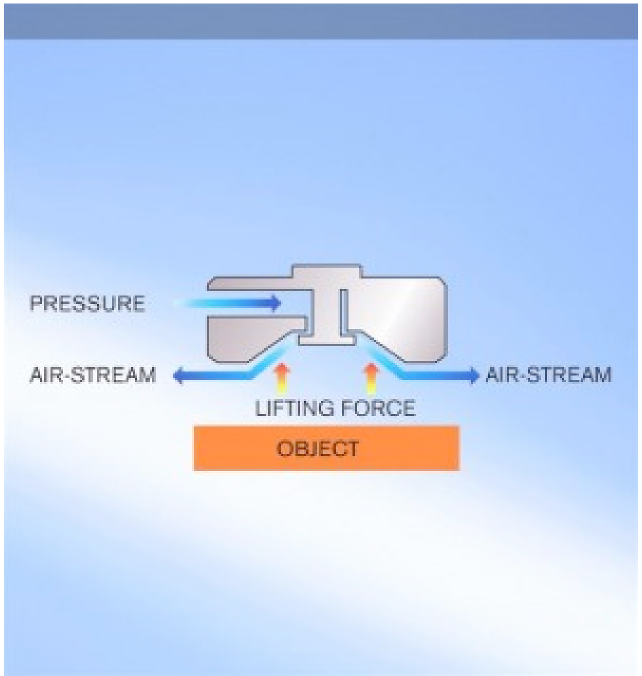
Applications:

- Lift from an airfoil (be careful of analysis)
- Carburetor design (venturi creates region of low pressure that draws in fuel and mixes it
- Pitot tube used to determine airspeed of an aircraft
- Allows sail-craft to move faster than the wind!





Bernoulli's Principle



Applications:

- Noncontact grippers (e.g., for solar cells which are easily contaminated)
- Windcube: Generator employing shroud that generates electricity with wind speeds as low as 5 mph

Easy Experiments:

- Blow between two ping-pong balls suspended from strings
- Collapse a paper house
- Suspend a balloon or ping-pong ball over a stream of air
- Blow air out of a funnel
- Other cool activities :)

Bernoulli's Principle

Applications:

- Curve ball