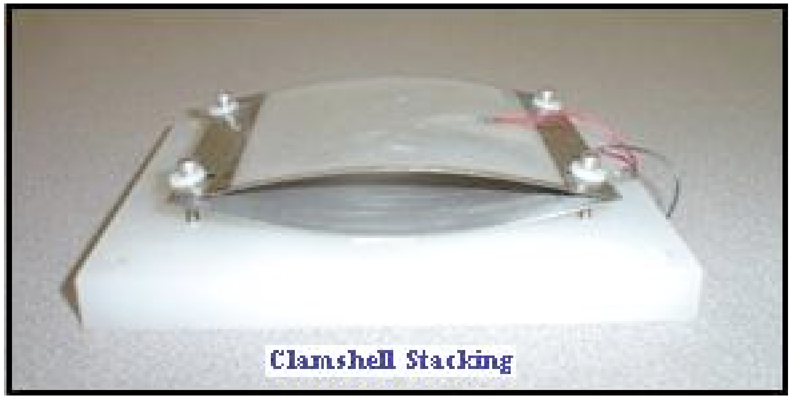
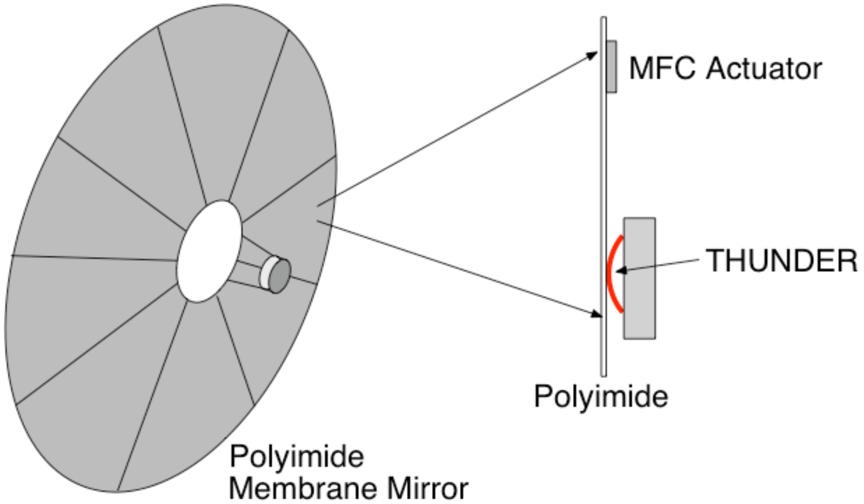
Thin Beam Models

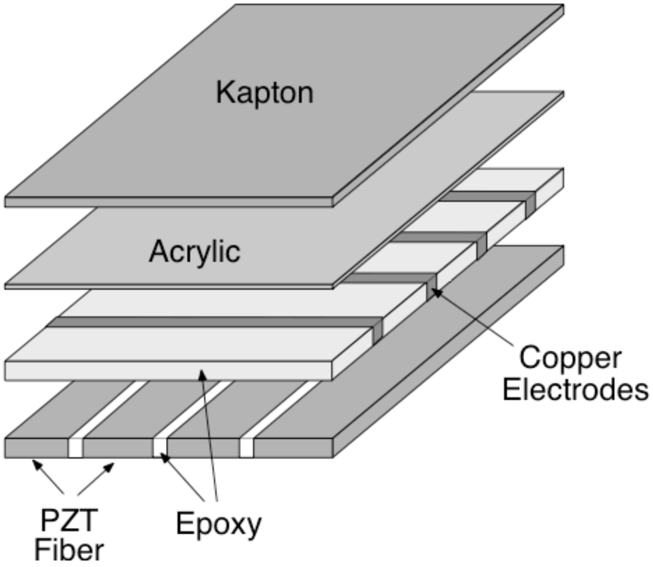
”He is a self-made man and worships his creator,” John Bright

Motivating Applications

THUNDER: **TH**in layer **UN**imorph **D**riv**ER** and sensor



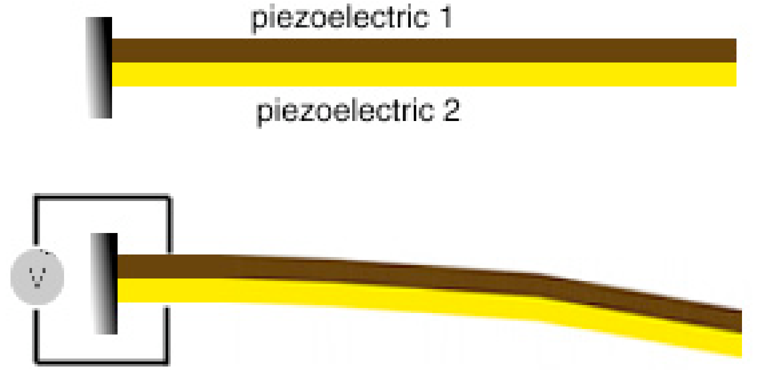




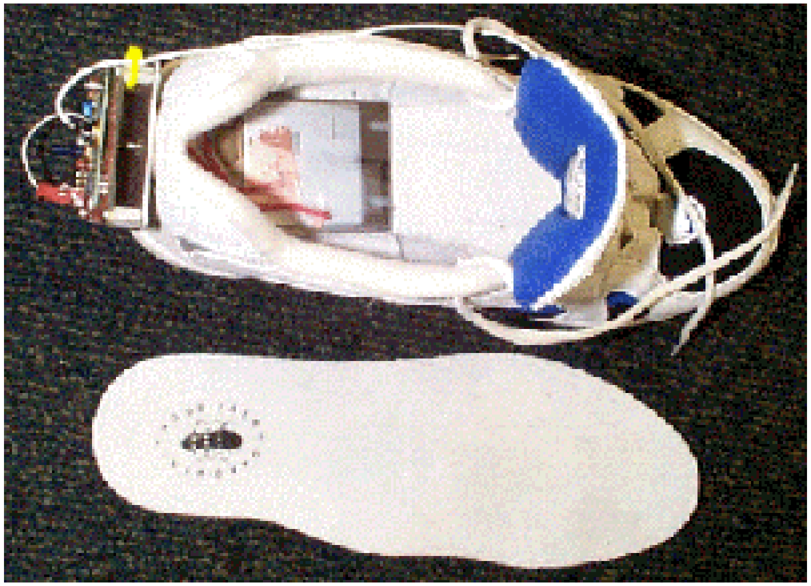
Motivating Applications

Unimorphs: e.g., actuation for micro-robotics

Figure 2. Bimorph



Energy Harvesting:



Modeling Assumptions

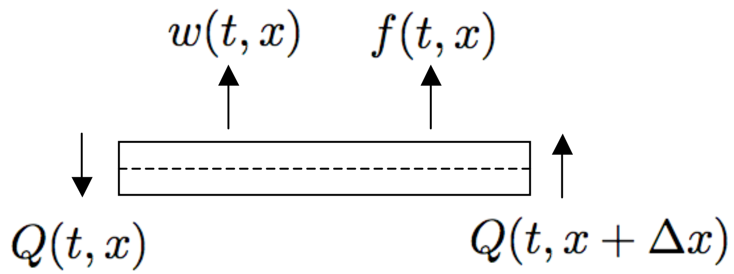
Physical Assumptions:

- Consider only motion in the x-direction
- Small displacements which permits linear theory
- Planar cross-sections remain planar during bending
- Negligible shear deformations (thickness is small compared with the length)

Force and Moment Balance

Notation:

- ρ: Density (kg/m^3)
- γ: Air damping coefficient
- Y: Young's modulus
- c: Kelvin-Voigt damping coefficient
- $Q(t,x)$: Shear resultant
- $M(t,x)$: Bending moment



Force Balance:

$$\int_x^{x+\Delta x} \rho h b \frac{\partial^2 w}{\partial t^2}(t,s)ds = Q(t,x+\Delta x) - Q(t,x) + \int_x^{x+\Delta x} f(t,s)ds - \gamma \int_x^{x+\Delta x} \frac{\partial w}{\partial t}(t,s)ds$$

$$\Rightarrow \rho h b \frac{\partial^2 w}{\partial t^2} + \gamma \frac{\partial w}{\partial t} = \frac{\partial Q}{\partial x} + f$$

Moment Balance:

$$M(t,x+\Delta x) - M(t,x) - Q(t,x+\Delta x)\Delta x - \int_x^{x+\Delta x} f(t,s)(s-x)dx = 0$$

$$\Rightarrow \frac{\partial M}{\partial x} = Q$$

Constitutive and Kinematic Relations

Force and Moment Balance:

$$\rho bh \frac{\partial^2 w}{\partial t^2} + \gamma \frac{\partial w}{\partial t} - \frac{\partial^2 M}{\partial x^2} = f$$

Constitutive and Kinematic Relations:

- Assumption 1: Negligible internal damping

  $$\sigma = Y\varepsilon$$

- Assumption 2: Stong internal damping

  $$\sigma = Y\varepsilon + c\dot{\varepsilon}$$

Assumption 1:

$$\varepsilon = \kappa(z - z_n) = \kappa z \quad \text{if symmetric}$$

$$\begin{aligned} M &= \int_{-h/2}^{h/2} Y\kappa z^2 dz \\ &= \kappa Y I \quad \text{where } I = \frac{bh^3}{12} \end{aligned}$$

Constitutive and Kinematic Relations

Note:

$$\kappa = -\frac{\partial^2 w}{\partial x^2}$$

Moment:

$$M = -YI\frac{\partial^2 w}{\partial x^2}$$

Model (Assumption 1):

$$\rho bh\frac{\partial^2 w}{\partial t^2} + \gamma\frac{\partial w}{\partial t} + \frac{\partial}{\partial x^2}\left(YI\frac{\partial^2 w}{\partial x^2}\right) = f$$

Model (Assumption 2):

$$\rho bh\frac{\partial^2 w}{\partial t^2} + \gamma\frac{\partial w}{\partial t} + \frac{\partial}{\partial x^2}\left(YI\frac{\partial^2 w}{\partial x^2} + cI\frac{\partial^3 w}{\partial x^2 \partial t}\right) = f$$

Boundary Conditions

1. Cantilever: Fixed at $x = 0$, free at $x = \ell$

$$w(t,0) = \frac{\partial w}{\partial x}(t,0) = 0$$

$$M(t,\ell) = Q(t,\ell) = 0$$

2. Pinned at Both Ends:

$$w(t,0) = M(t,0) = 0$$

$$w(t,\ell) = M(t,\ell) = 0$$

Weak Formulation -- Cantilever Beam

State Space:

$$X = L^2(0,\ell)$$

$$\langle \psi, \phi \rangle_X = \int_0^\ell \rho h b \psi \phi dx$$

Space of Test Functions: Cantilever beam

$$V = H_0^2(0,\ell) = \{\phi \in H^2(0,\ell) \,|\, \phi(0) = \phi'(0) = 0\}$$

$$\langle \psi, \phi \rangle_V = \int_0^\ell YI \psi'' \phi'' dx$$

Weak Formulation: Multiply by test functions and integrate by parts to get

$$\int_0^\ell \rho h b \frac{\partial^2 w}{\partial t^2} \phi dx + \gamma \int_0^\ell \frac{\partial w}{\partial t} \phi dx - \int_0^\ell M \frac{d^2 \phi}{dx^2} dx = \int_0^\ell f \phi dx$$

$$\Rightarrow \int_0^\ell \rho h b \frac{\partial^2 w}{\partial t^2} \phi dx + \gamma \int_0^\ell \frac{\partial w}{\partial t} \phi dx + \int_0^\ell \left[YI \frac{\partial^2 w}{\partial x^2} + cI \frac{\partial^3 w}{\partial x^2 \partial t} \right] \frac{d^2 \phi}{dx^2} dx = \int_0^\ell f \phi dx$$

Weak Formulation -- Cantilever Beam

Hamiltonian Formulation: Kinetic and potential energies

$$K = \frac{1}{2}\int_0^\ell \rho h b w_t^2(t,x)dx$$

$$U = \frac{1}{2}\int_0^\ell YIw_{xx}^2(t,x)dx$$

Action Integral:

$$\mathcal{A} = \int_{t_0}^{t_1}[K-U]dt$$

Analytic Beam Model Solution

Consider:

$$\rho \frac{\partial^2 w}{\partial t^2} + YI \frac{\partial^4 w}{\partial x^4} = 0$$

Separation of Variables:

$$w(t, x) = X(x)T(t)$$

$$\Rightarrow \rho \frac{\ddot{T}(t)}{T(t)} + YI \frac{X''''(x)}{X(x)} = 0$$

Thus

$$\ddot{T}(t) + \frac{\alpha}{\rho} T(t) = 0 \qquad\qquad YIX''''(x) = \alpha X(x)$$

Initial Conditions $\qquad\qquad$ Boundary Conditions

General Solutions: $\Omega = \sqrt{\frac{\alpha}{\rho}} \quad , \quad \xi = \left(\frac{\alpha}{YI}\right)^{1/4}$

$$T(t) = A\cos(\Omega t) + B\sin(\Omega t)$$

$$X(x) = a\cos(\xi x) + b\cosh(\xi x) + c\sin(\xi x) + d\sinh(\xi x)$$

Analytic Beam Model Solution

Pinned End Conditions:

$$w(0,t) = w''(0,t) = w(\ell,t) = w''(\ell,t) = 0$$

$$X(0) = X''(0) = X(\ell) = X''(\ell) = 0$$

Thus

$$a = b = d = 0$$

and

$$\xi = \left(\frac{\alpha}{YI}\right)^{1/4} = \frac{n\pi}{\ell}$$

$$\Rightarrow \alpha_n = YI\left(\frac{n\pi}{\ell}\right)^4$$

Eigenfunctions and Frequencies:

$$X_n(x) = \sin\left(\frac{n\pi x}{\ell}\right) \quad , \quad \Omega_n = \sqrt{\frac{YI}{\rho}\frac{n^2\pi^2}{\ell^2}}$$

Analytic Beam Model Solution

Cantilever End Conditions:

$$w(0,t) = w'(0,t) = w''(\ell,t) = w'''(\ell,t) = 0$$

$$X(0) = X'(0) = X''(\ell) = X'''(\ell) = 0$$

Here

$$a = -b \;, c = -d$$

$$\begin{bmatrix} \cos(\xi\ell) + \cosh(\xi\ell) & \sin(\xi\ell) + \sinh(\xi\ell) \\ -\sin(\xi\ell) + \sinh(\xi\ell) & \cos(\xi\ell) + \cosh(\xi\ell) \end{bmatrix} \begin{bmatrix} b \\ d \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \end{bmatrix}$$

Strategy: Numerically determine ξ_n that yield $|A| = 0$

Galerkin Method for Beam Model

Weak Formulation: See Section 8.3 of Smith, 2005

$$\int_0^\ell \rho \frac{\partial^2 w}{\partial t^2}\phi\, dx + \int_0^\ell \gamma \frac{\partial w}{\partial t}\phi\, dx + \int_0^\ell YI \frac{\partial^2 w}{\partial x^2}\frac{d^2\phi}{dx^2}\, dx$$

$$+ \int_0^\ell cI \frac{\partial^3 w}{\partial x^2 \partial t}\frac{d^2\phi}{dx^2}\, dx = \int_0^\ell f\phi\, dx + \int_0^\ell k_p V(t)\frac{d^2\phi}{dx^2}\, dx$$

for all $\phi \in V = H_0^2(0,\ell)$

Basis:

$$\phi_j(x) = \begin{cases} \widehat{\phi}_0(x) - 2\widehat{\phi}_{-1}(x) - 2\widehat{\phi}_1(x) & , \quad j = 1 \\ \widehat{\phi}_j(x) & , \quad j = 2, \ldots, N+1 \end{cases}$$

where

$$\widehat{\phi}_j(x) = \frac{1}{h^3}\begin{cases} (x - x_{j-2})^3 \,, & x \in [x_{j-2}, x_{j-1}) \\ h^3 + 3h^2(x - x_{j-1}) + 3h(x - x_{j-1})^2 - 3(x - x_{j-1})^3 \,, & x \in [x_{j-1}, x_j) \\ h^3 + 3h^2(x_{j+1} - x) + 3h(x_{j+1} - x)^2 - 3(x_{j+1} - x)^3 \,, & x \in [x_j, x_{j+1}) \\ (x_{j+2} - x)^3 \,, & x \in [x_{j+1}, x_{j+2}) \\ 0 \,, & \text{otherwise} \end{cases}$$

Galerkin Method for Beam Model

Approximate Solution:

$$w^N(t,x) = \sum_{j=1}^{N+1} w_j(t)\phi_j(x)$$

System:

$$\mathbb{M}\,\ddot{\mathbf{w}} + \mathbb{Q}\,\dot{\mathbf{w}} + \mathbb{K}\,\mathbf{w} = \mathbf{f} + V(t)\mathbf{b}$$

where

$$\mathbf{w}(t) = [w_1(t)\,,\,\ldots\,,\,w_{N+1}(t)]^T$$

and

$$[\mathbb{M}]_{ij} = \int_0^\ell \rho\phi_i\phi_j dx$$

$$[\mathbb{Q}]_{ij} = \int_0^\ell \left[\gamma\phi_i\phi_j + cI\phi_i''\phi_j''\right] dx$$

$$[\mathbb{K}]_{ij} = \int_0^\ell YI\phi_i''\phi_j'' dx$$

$$[\mathbf{f}]_i = \int_0^\ell f\phi_i dx \quad , \quad [\mathbf{b}]_i = \int_0^\ell \phi_i'' dx$$