Acoustic Models: Analytic Solution Techniques

"He suffered from paralysis by analysis," Harold S. Geneen

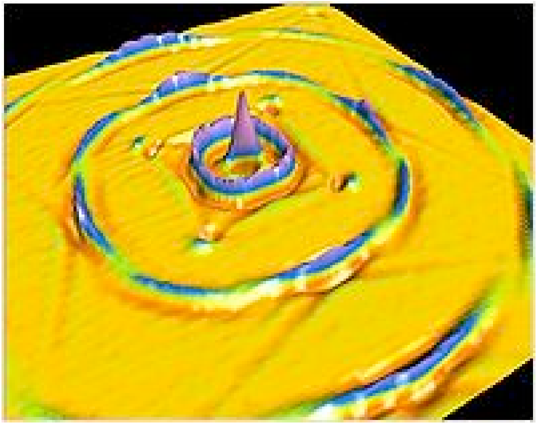
Linear Wave Equations for Sound

Pressure:

$$\frac{\partial^2 \hat{p}}{\partial t^2} = c^2 \Delta \hat{p}$$

Boundary Conditions

Initial Conditions



Velocity:

$$\frac{\partial^2 \hat{u}}{\partial t^2} = c^2 \Delta \hat{u}$$

Boundary Conditions

Initial Conditions

Potential:

$$\frac{\partial^2 \phi}{\partial t^2} = c^2 \Delta \phi$$

Boundary Conditions

Initial Conditions

D'Alembert's Solution of Wave Equation

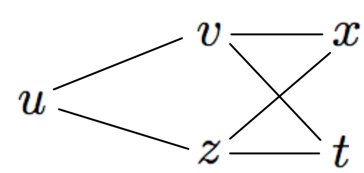
Model:

$$\frac{\partial^2 u}{\partial t^2} = c^2 \frac{\partial^2 u}{\partial x^2} \qquad -\infty < x < \infty \; , \; t > 0$$

$$u(0,x) = f(x) \; , \; u_t(0,x) = g(x) \qquad -\infty < x < \infty$$

New Independent Variables:

$$v = x + ct \; , \; z = x - ct$$



Thus

$$u_x = u_v v_x + u_z z_x = u_v + u_z$$

and

$$\begin{aligned} u_{xx} &= (u_v + u_z)_x \\ &= (u_v + u_z)_v v_x + (u_v + u_z)_z z_x \\ &= u_{vv} + 2u_{vz} + u_{zz} \end{aligned}$$

Similarly,

$$u_{tt} = c^2(u_{vv} - 2u_{vz} + u_{zz})$$

D'Alembert's Solution of Wave Equation

Transformed Differential Equation:

$$u_{vz} \equiv \frac{\partial^2 u}{\partial v \partial z} = 0$$

$$\Rightarrow \frac{\partial u}{\partial v} = h(v)$$

$$\Rightarrow u(v, z) = \int_0^v h(s) ds + B(z)$$

$$\Rightarrow u(v, z) = A(v) + B(z)$$

where $A(v)$ and $B(z)$ are arbitrary functions that satisfy initial conditions.

General Solution:

$$u(t, x) = A(x + ct) + B(x - ct)$$

D'Alembert's Solution of Wave Equation

Initial Conditions:

$$u(0,x) = A(x) + B(x) = f(x) \quad , \quad u_t(0,x) = cA'(x) - cB'(x) = g(x)$$

Second equation yields

$$A(x) - B(x) = \frac{1}{c}\int_0^x g(s)ds + D$$

$$\Rightarrow \quad A(x) = \frac{1}{2}f(x) + \frac{1}{2c}\int_0^x g(s)ds + \frac{D}{2}$$

$$B(x) = \frac{1}{2}f(x) - \frac{1}{2c}\int_0^x g(s)ds - \frac{D}{2}$$

D'Alembert's Solution:

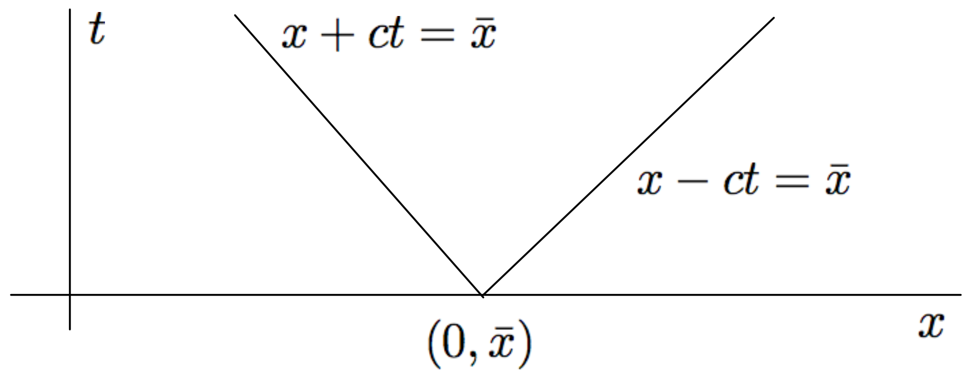
$$u(t,x) = \frac{1}{2}\left[f(x+ct) + f(x-ct)\right] + \frac{1}{2c}\int_{x-ct}^{x+ct} g(s)ds$$

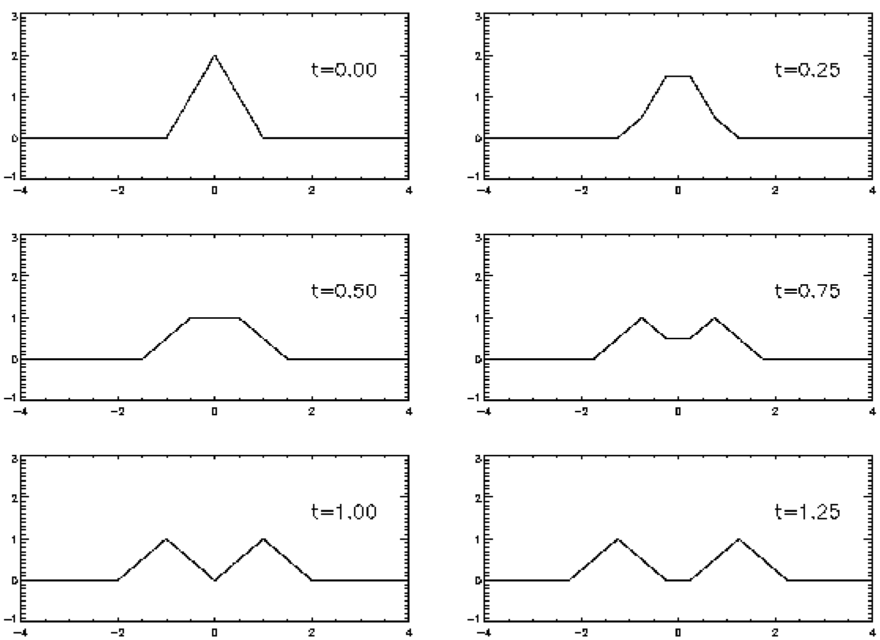
D'Alembert's Solution of Wave Equation

Physical Interpretation: Consider an infinite string with local initial conditions

Characteristics: Curves

$$x + ct = \text{constant}$$
$$x - ct = \text{constant}$$





Properties of Fourier Series

Fourier Series: Consider the representation

$$f(x) = \frac{a_0}{2} + \sum_{n=1}^{\infty}\left[a_n \cos\left(\frac{n\pi x}{L}\right) + b_n \sin\left(\frac{n\pi x}{L}\right)\right]$$

for a $2L$ periodic, continuous function $f(x)$. If the series converges uniformly, it is the Fourier series for $f(x)$.

Note:

$$\int_{-L}^{L} f(x)dx = a_0 L$$

$$\int_{-L}^{L} \sin\left(\frac{m\pi x}{L}\right)\sin\left(\frac{n\pi x}{L}\right)dx = \int_{-L}^{L} \cos\left(\frac{m\pi x}{L}\right)\cos\left(\frac{n\pi x}{L}\right)dx = 0 \quad \text{for } m \neq n$$

$$\int_{-L}^{L} \sin\left(\frac{m\pi x}{L}\right)\cos\left(\frac{n\pi x}{L}\right)dx = 0 \quad \text{for all } m, n$$

$$\int_{-L}^{L} \sin^2\left(\frac{m\pi x}{L}\right)dx = \cos^2\left(\frac{n\pi x}{L}\right)dx = L \quad \text{for } n \geq 1$$

Properties of Fourier Series

Fourier Coefficients:

$$a_n = \frac{1}{L}\int_{-L}^{L} f(x)\cos\left(\frac{n\pi x}{L}\right)dx \quad \text{for } n = 0, 1, 2, \ldots$$

$$b_n = \frac{1}{L}\int_{-L}^{L} f(x)\sin\left(\frac{n\pi x}{L}\right)dx \quad \text{for } n = 1, 2, \ldots$$

Theorem: Suppose that f and f' are piecewise continuous on the interval $-L \leq x \leq L$. Further, suppose that f is defined outside the interval $-L \leq x \leq L$ so that it is periodic with period $2L$. Then f has a Fourier series

$$f(x) = \frac{a_0}{2} + \sum_{n=1}^{\infty}\left[a_n\cos\left(\frac{n\pi x}{L}\right) + b_n\sin\left(\frac{n\pi x}{L}\right)\right]$$

with the above coefficients. The Fourier series converges to $f(x)$ at all points where f is continuous, and to $[f(x+) + f(x-)]/2$ at all points where f is discontinuous (see Boyce and DiPrima).

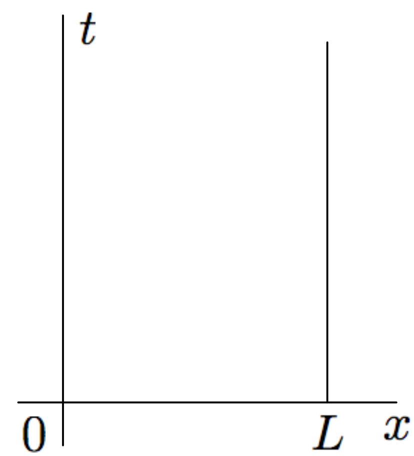
Separation of Variables

Model:

$$u_{tt} = c^2 u_{xx} \quad , \quad 0 < x < L \;, t > 0$$

$$u(0,x) = f(x) \;,\; u_t(0,x) = g(x) \quad ,0 \le x \le L$$

$$u(t,0) = u(t,L) = 0 \quad ,t \ge 0$$



Separation of Variables: Consider solutions of the form

$$u(t,x) = X(x)T(t)$$

$$\Rightarrow X(x)\ddot{T}(t) = c^2 X''(x)T(t)$$

$$\Rightarrow \frac{X''(x)}{X(x)} = \frac{\ddot{T}(t)}{c^2 T(t)} = K$$

Consider first

$$X''(x) - KX(x) = 0$$

$$X(0) = X(L) = 0$$

Separation of Variables

Note:

$$\int_0^L \left[XX'' - KX^2\right] dx = \int_0^L \left[-\left(X'\right)^2 - KX^2\right] dx = 0$$

$$\Rightarrow \int_0^L [X'(x)^2 + KX^2(x)]dx = 0$$

If $K \geq 0$, this implies $X(x) = K = 0$. Thus $K < 0$ so take $K = -\lambda^2, \lambda > 0$.

Boundary Value Problem: Helmholtz Equation

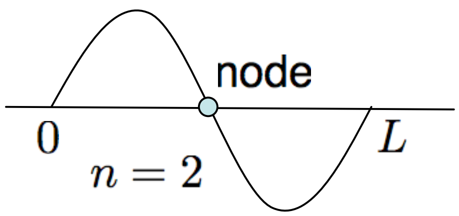
$$X''(x) + \lambda^2 X(x) = 0$$

$$X(0) = X(L) = 0$$

Solution: $X(x) = A\cos(\lambda x) + B\sin(\lambda x)$

$$X(0) = 0 \Rightarrow A = 0$$

$$X(L) = 0 \Rightarrow \lambda L = n\pi$$



Thus $X_n(x) = B_n \sin(\lambda_n x)$, $\lambda_n = \frac{n\pi}{L}$, $B_n \neq 0$

Modes or Eigenfunctions

Separation of Variables

Initial Value Problem Solution:

$$T_n(t) = C_n \cos(\lambda_n ct) + D_n \sin(\lambda_n ct)$$

General Solution to PDE:

$$u(t,x) = \sum_{n=1}^{\infty} [a_n \cos(\lambda_n ct) + b_n \sin(\lambda_n ct)] \sin(\lambda_n x)$$

Initial Conditions:

$$f(x) = \sum_{n=1}^{\infty} a_n \sin\left(\frac{n\pi x}{L}\right) \quad , \quad g(x) = \sum_{n=1}^{\infty} b_n \lambda_n c \sin\left(\frac{n\pi x}{L}\right)$$

$$\Rightarrow \quad a_n = \frac{2}{L}\int_0^L f(x)\sin\left(\frac{n\pi x}{L}\right)dx$$

$$b_n = \frac{2}{\lambda_n cL}\int_0^L g(x)\sin\left(\frac{n\pi x}{L}\right)dx$$

Example

Model: Take $g(x) = 0$ so $b_n = 0$

Note:

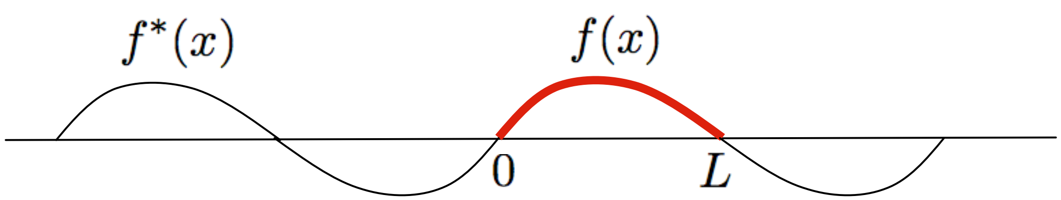
$$\cos(\lambda_n ct)\sin(\lambda_n x) = \frac{1}{2}[\sin(\lambda_n(x-ct)) + \sin(\lambda_n(x+ct))]$$

Solution:

$$u(t,x) = \frac{1}{2}\sum_{n=1}^{\infty} a_n \sin(\lambda_n(x-ct)) + \frac{1}{2}\sum_{n=1}^{\infty} a_n \sin(\lambda_n(x+ct))$$

$$\Rightarrow u(t,x) = \frac{1}{2}[f^*(x-ct) + f^*(x+ct)]$$

where f^* is the odd periodic extension of f with period $2L$.



Relationship Between Wavelength and Frequency

Note:

- Vibrate structure with frequency f for time t
- Generate $N = ft$ waves
- First wave travels distance ct
- Wavelength is ratio of distance to number of waves in this distance

Relationship:

$$\lambda = \frac{ct}{N} = \frac{ct}{ft}$$

$$\Rightarrow \lambda = \frac{c}{f}$$