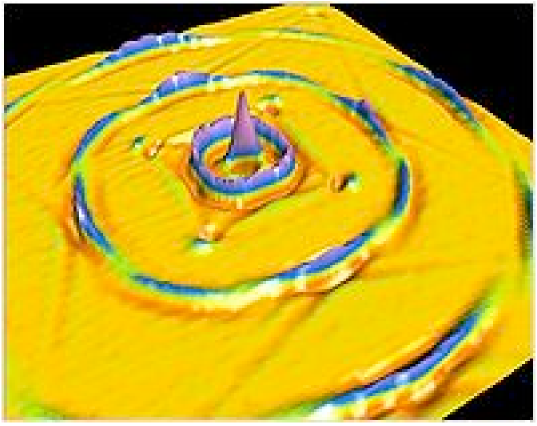
Acoustic Models: Numerical Solution Techniques

“First get your facts; then you can distort them at your leasure.” Mark Twain

Linear Wave Equations for Sound



Pressure:

$$\frac{\partial^2 \hat{p}}{\partial t^2} = c^2 \Delta \hat{p}$$

Boundary Conditions

Initial Conditions

Velocity:

$$\frac{\partial^2 \hat{u}}{\partial t^2} = c^2 \Delta \hat{u}$$

Boundary Conditions

Initial Conditions

Potential:

$$\frac{\partial^2 \phi}{\partial t^2} = c^2 \Delta \phi$$

Boundary Conditions

Initial Conditions

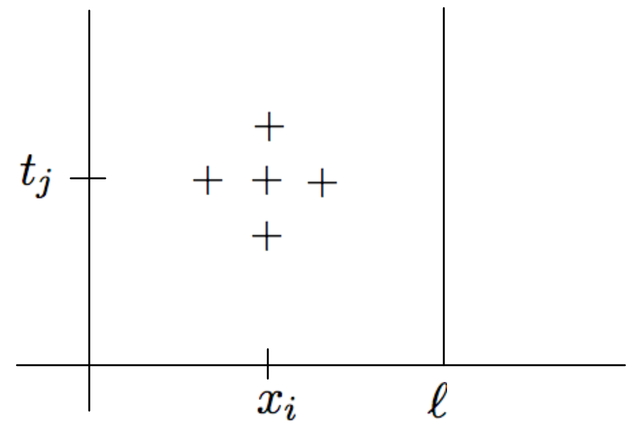
Finite Difference Methods

Consider

$$u_{tt} - c^2 u_{xx} = F$$

$$u(t,0) = u(t,\ell) = 0$$

$$u(0,x) = f(x)\ ,\ u_t(0,x) = g(x)$$



Grid: $\Delta = \{(x_i, t_j) \,|\, x_i = ih, t_j = jk\}$

System: $m = c\dfrac{k}{h}$

$$\frac{1}{k^2}\left[u_{i,j-1} - 2u_{i,j} + u_{i,j+1}\right] - \frac{c^2}{h^2}\left[u_{i-1,j} - 2u_{i,j} + u_{i+1,j}\right] = F(x_i, t_j)$$

$$\Rightarrow u_{i,j+1} = m^2\left[u_{i+1,j} + u_{i-1,j}\right] + 2(1-m^2)u_{i,j} - u_{i,j-1} + k^2 F(x_i, t_j)$$

Initial Conditions:

$$g(x_i) \approx \frac{u_{i,1} - u_{i,-1}}{2k} \Rightarrow u_{i,-1} = u_{i,1} - 2kg(x_i)$$

$$\Rightarrow u_{i,1} = \frac{m^2}{2}[f(x_{i+1}) + f(x_{i-1})] + (1-m^2)f(x_i) + kg(x_i) + \frac{k^2}{2}F(x_i, t_0)$$

Finite Difference Methods

Matrix System:

$$\begin{bmatrix} u_{1,j+1} \\ u_{2,j+1} \\ u_{3,j+1} \\ \\ \vdots \\ u_{n-1,j+1} \end{bmatrix} = \begin{bmatrix} 2(1-m^2) & m^2 & 0 & & \\ m^2 & 2(1-m^2) & m^2 & & \\ & & & & \\ & \ddots & \ddots & \ddots & \\ & & & & \\ & 0 & m^2 & 2(1-m^2) \end{bmatrix} \begin{bmatrix} u_{1,j} \\ u_{2,j} \\ u_{3,j} \\ \\ \vdots \\ u_{n-1,j} \end{bmatrix} - \begin{bmatrix} u_{1,j-1} \\ u_{2,j-1} \\ u_{3,j-1} \\ \\ \vdots \\ u_{n-1,j-1} \end{bmatrix} + k^2 \begin{bmatrix} F(x_1,t_j) \\ F(x_2,t_j) \\ F(x_3,t_j) \\ \\ \vdots \\ F(x_{n-1},t_j) \end{bmatrix}$$

Local Truncation Error: $\mathcal{O}(h^2) + \mathcal{O}(k^2)$

Stability Requirement: $k \leq \frac{1}{c}h$

Option to Eliminate Stability Constraint: Implicit time differencing

Finite Element Methods

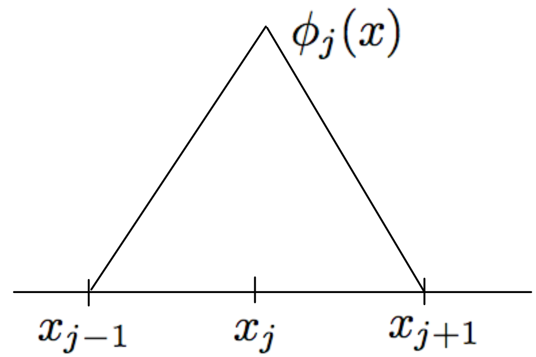
Weak Formulation:

$$\int_0^\ell \frac{\partial^2 u}{\partial t^2}\phi dx + c^2 \int_0^\ell \frac{\partial u}{\partial x}\frac{d\phi}{dx} dx = \int_0^\ell F\phi dx$$

for all $\phi \in V = H_0^1(0,\ell) = \{\phi \in H^1(0,\ell) \,|\, \phi(0) = \phi(\ell) = 0\}$

Linear Basis:

$$\phi_j(x) = \frac{1}{h}\begin{cases} x - x_{j-1}\,, & x_{j-1} \le x < x_j \\ x_{j+1} - x\,, & x_j \le x \le x_{j+1} \\ 0\,, & \text{otherwise} \end{cases}$$



Approximate Solution:

$$u^N(t,x) = \sum_{j=1}^{N-1} u_j(t)\phi_j(x)$$

System: For $i = 1, \cdots, N-1$

$$\sum_{j=1}^{N-1} \ddot{u}_j(t) \int_0^\ell \phi_i\phi_j dx + \sum_{j=1}^{N-1} u_j(t) \int_0^\ell c^2 \phi_i'\phi_j' dx = \int_0^\ell F\phi_i dx$$

Finite Element Methods

Matrix System:

$$\begin{bmatrix} \int_0^L \phi_1\phi_1 dx & \cdots & \int_0^L \phi_1\phi_{N-1} dx \\ \vdots & & \vdots \\ \int_0^L \phi_{N-1}\phi_1 dx & \cdots & \int_0^L \phi_{N-1}\phi_{N-1} dx \end{bmatrix} \begin{bmatrix} \ddot{u}_1(t) \\ \vdots \\ \ddot{u}_{N-1}(t) \end{bmatrix}$$

$$+ c^2 \begin{bmatrix} \int_0^L \phi_1'\phi_1' dx & \cdots & \int_0^L \phi_1'\phi_{N-1}' dx \\ \vdots & & \vdots \\ \int_0^L \phi_{N-1}'\phi_1' dx & \cdots & \int_0^L \phi_{N-1}'\phi_{N-1}' dx \end{bmatrix} \begin{bmatrix} u_1(t) \\ \vdots \\ u_{N-1}(t) \end{bmatrix} = \begin{bmatrix} \int_0^\ell F\phi_1 dx \\ \vdots \\ \int_0^\ell F\phi_{N-1} dx \end{bmatrix}$$

2nd-Order System:

$$\mathbb{M}\ddot{\mathbf{u}}(t) + \mathbb{K}\mathbf{u}(t) = \mathbf{f}(t)$$

Matrices:

$$\mathbb{M} = h \begin{bmatrix} \frac{2}{3} & \frac{1}{6} & 0 & \cdots & 0 \\ \frac{1}{6} & \frac{2}{3} & \frac{1}{6} & & \\ & \ddots & \ddots & \ddots & \\ & & \frac{1}{6} & \frac{2}{3} & \frac{1}{6} \\ 0 & \cdots & 0 & \frac{1}{6} & \frac{2}{3} \end{bmatrix}, \quad \mathbb{K} = \frac{c^2}{h} \begin{bmatrix} 2 & -1 & 0 & \cdots & 0 \\ -1 & 2 & -1 & & \\ & \ddots & \ddots & \ddots & \\ & & -1 & 2 & -1 \\ 0 & \cdots & 0 & -1 & 2 \end{bmatrix}$$

Finite Element Methods

1st-Order System:

$$\dot{\mathbf{z}}(t) = \mathbb{A}\,\mathbf{z}(t) + \mathbf{F}(t)$$

$$\mathbf{z}(0) = \mathbf{z}_0$$

where

$$\mathbb{A} = \begin{bmatrix} 0 & \mathbb{I} \\ -\mathbb{M}^{-1}\mathbb{K} & 0 \end{bmatrix} \quad , \quad \mathbf{F}(t) = \begin{bmatrix} 0 \\ \mathbb{M}^{-1}\mathbf{f}(t) \end{bmatrix}$$

Commercial Packages

PDE Toolbox in MATLAB:

- General hyperbolic equation

$$d\frac{\partial^2 u}{\partial t^2} - \nabla \cdot (c\nabla u) + au = f$$

- Helmholtz equation

$$\nabla^2 u + k^2 u = 0$$

where k is the wave number