Acoustic Models

“It's easy to play any musical instrument: all you have to do is touch the right key at the right time and the instrument will play itself,” J.S. Bach

Motivating Applications

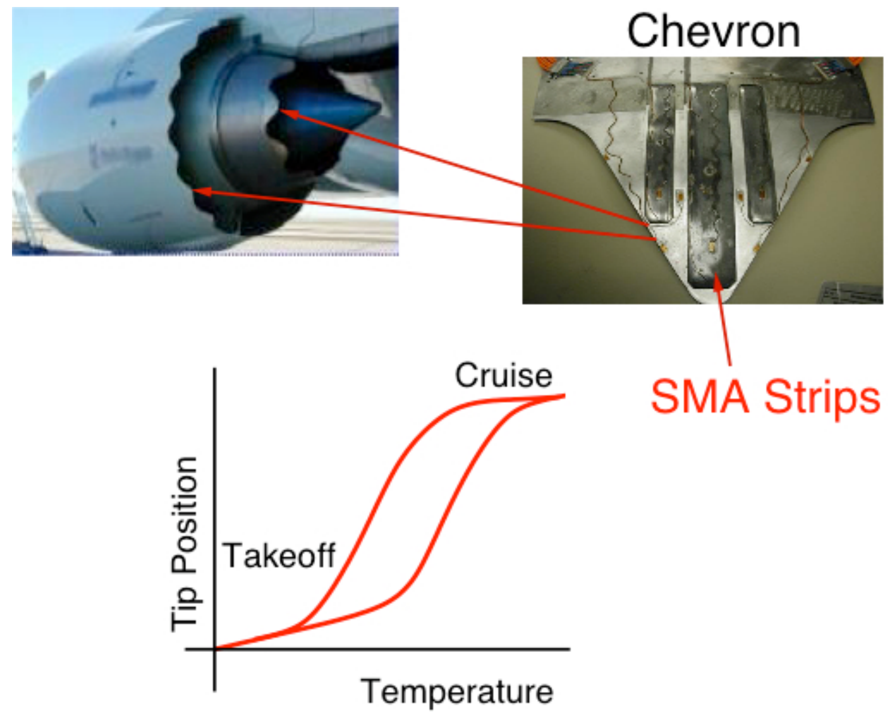
Jet Noise Reduction:

• Employ chevrons to improve mixing and decrease jet noise

• Boeing experiments demonstrate 4 dB noise reduction

• 3 dB reduction if 1 of 2 engines turned off

• Influences aeroacoustic coupling and requires 2-D and 3-D SMA models



Reference: F.T. Calkins and J.H. Mabe, “Variable geometry jet nozzle using shape memory alloy actuators”

Motivating Applications

Duct Noise: Use secondary sources or structure-borne actuators to reduce duct or fan noise

Noise Canceling Headphones: e.g., cockpit noise for pilots is a significant problem for both health and communication

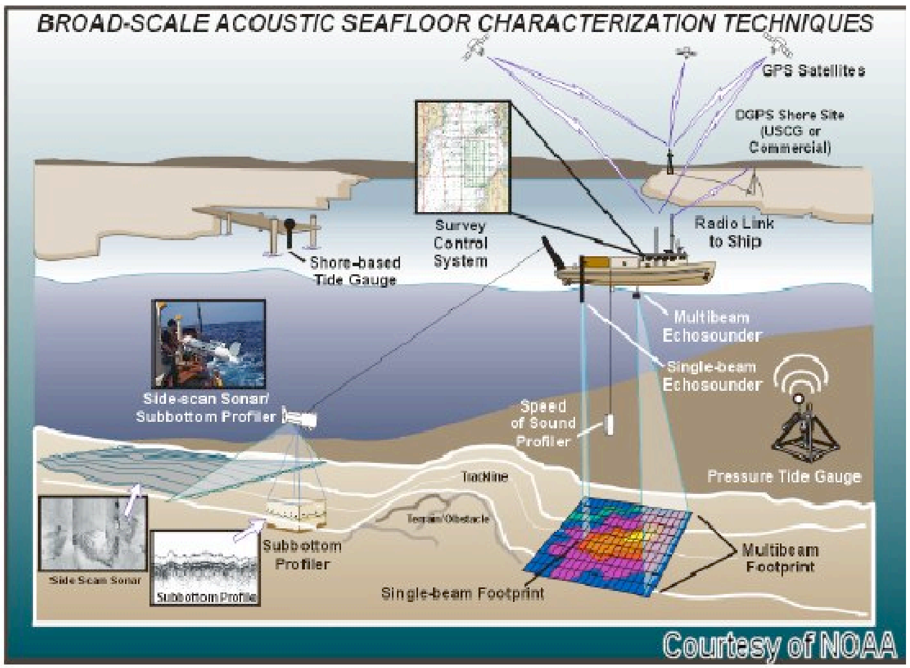
Motivating Applications

Improved Instrument and Amplifier Design: What makes a Stradivarius unique?

Remote Acoustic Sensing: Can we listen to a conversation using a laser?

Motivating Applications

Seismology and Acoustic Medical Imaging: e.g., ultrasound



Motivating Applications

Medical Treatment Strategies:

- Ultrasound to break up kidney stones
- Acoustic barriers for EM treatment
- Ultrasonic dental tools

Motivating Applications

SAW (Surface Acoustic Wave) Devices:

• Used in filters, oscillators and transformers; e.g., bandpass filters in cell phones

• Relies on transduction capabilities of PZT

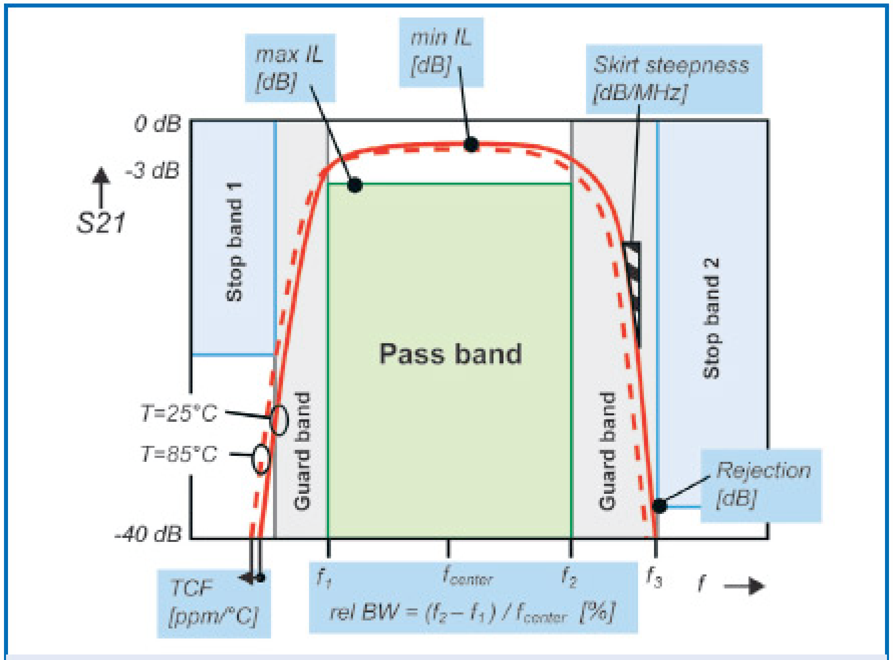
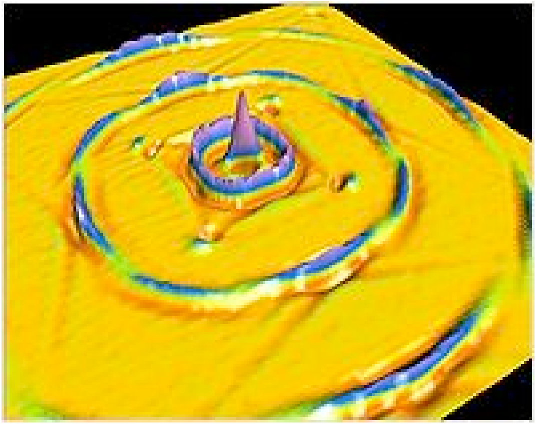
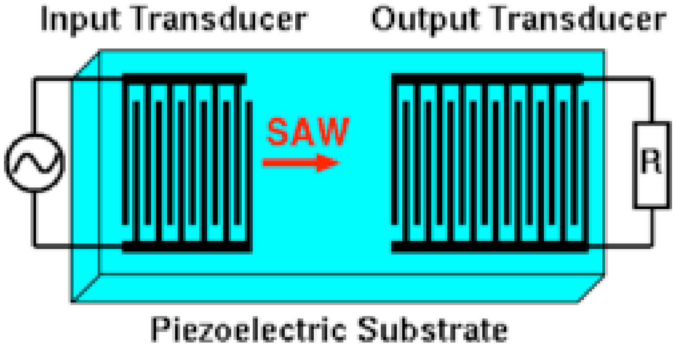


Figure 1: Illustration of RF Filter performance characteristics



Experimental image of SAW on a crystal of tellurium oxide.



Basic Definitions

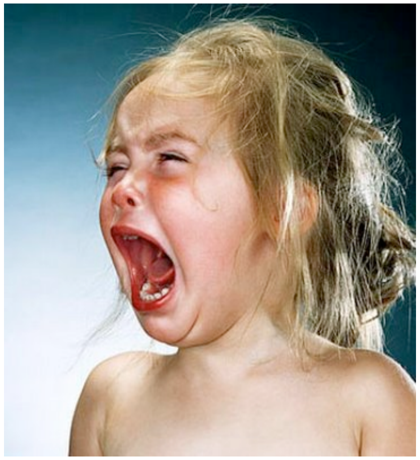
Acoustics: The science of sound including its production, transmission, and effects.

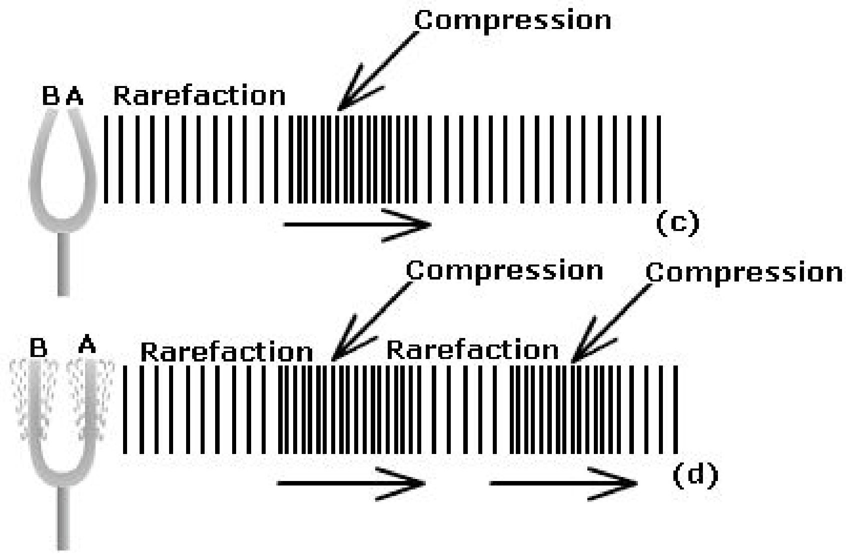
Sound: A traveling wave created by a vibrating object and propagated through a medium (gas, liquid, or solid) due to particle interactions.

• Because it is due to particle interactions, it is a mechanical wave. Thus sound cannot travel through a vacuum --- *"In space, no one can hear you scream," Alien 1979*

• Particle interactions yield oscillations in pressure resulting in local regions of compression and rarefaction.

Noise: Unwanted sound.

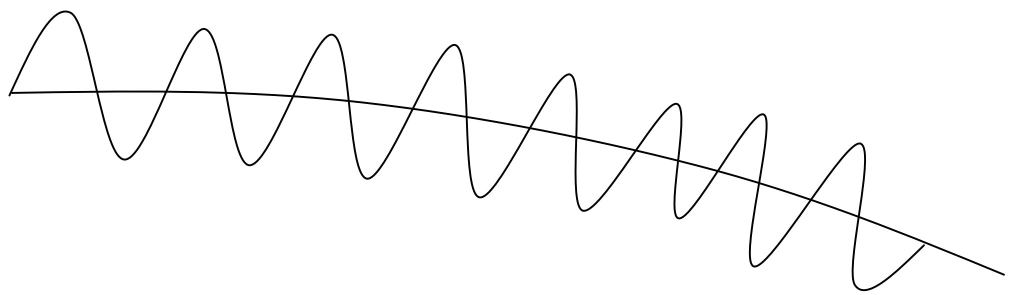




Sound Pressure and Acoustic Units

Acoustic or Sound Pressure: Difference between average local pressure of medium and pressure within sound wave at same point and time.

- Think about difference between changing pressure in an airplane (medium) and conversation (sound wave).



Instantaneous Sound Pressure: $p(t)$ Units: Pascals (N/m^2)

RMS Sound Pressure:

$$p_{rms} = \sqrt{\frac{1}{t_f - t_0} \int_{t_0}^{t_f} [p(t)]^2 dt} \qquad \text{Continuous Time}$$

$$p_{rms} = \sqrt{\frac{1}{N_t} \sum_{i=1}^{N_t} [p(t_i)]^2} \qquad \text{Discrete Time}$$

Sound Pressure Levels

Sound Pressure Level: Employs logarithmic scale

$$SPL = 10\log_{10}\left(\frac{p_{rms}^2}{p_{ref}^2}\right) = 20\log_{10}\left(\frac{p_{rms}}{p_{ref}}\right)$$

$$\text{Air:} \quad p_{ref} = 20\ \mu\text{Pa} = 2 \times 10^{-5}\ \text{N/m}^2$$

$$\text{Water:} \quad p_{ref} = 1\ \mu\text{Pa}$$

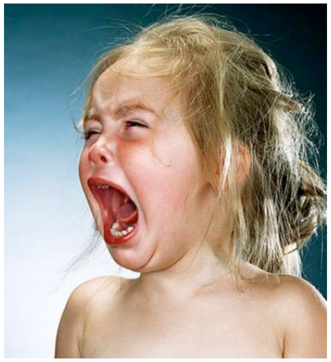
Nondimensional Units: Decibels (dB)

Examples of Sound Pressure and Sound Pressure Levels

Sound Source	RMS Sound Pressure	Sound Pressure Level
	Pa	dB re 20 microPA
Theoretical limit for undistorted sound	101,325	191
1883 Krakatoa eruption		Approx 180 at 100 miles
Stun grenades		170-180
Rocket launch		Approx 165
Hearing damage (short term exposure)	20	Approx 120
Jet engine at 100 m	6-200	110-140
Jackhammer/Disco	2	Approx 100
Hearing damage (long term exposure)	0.6	Approx 85
Normal Talking	0.002-0.02	40-60
Quiet rustling leaves	0.00006	10

Acoustic Sources

Direct Source:



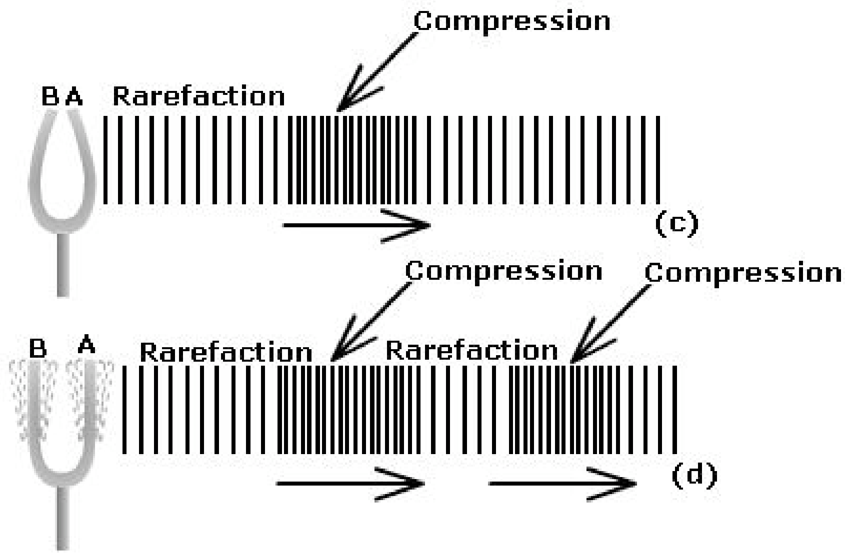
Note: Nature of sources affects control strategies including choice of actuators and sensors.

Structural-Acoustic Coupling:

Aeroacoustic Coupling:

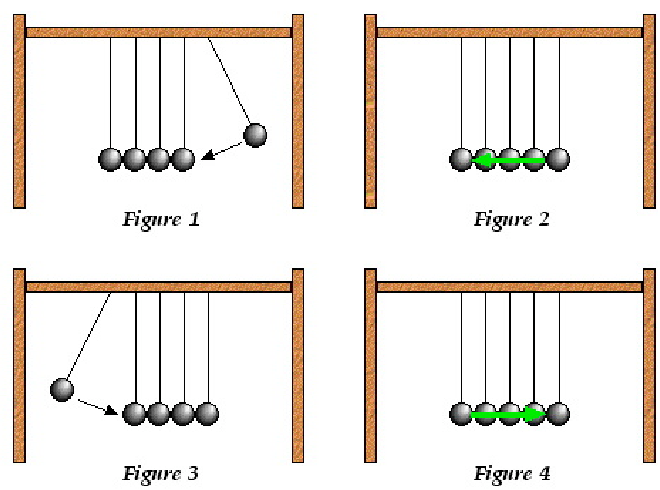
Physical Phenomena to be Modeled

Recall: Particle interactions yield oscillations in pressure resulting in local regions of compression and rarefaction.



Mechanisms:

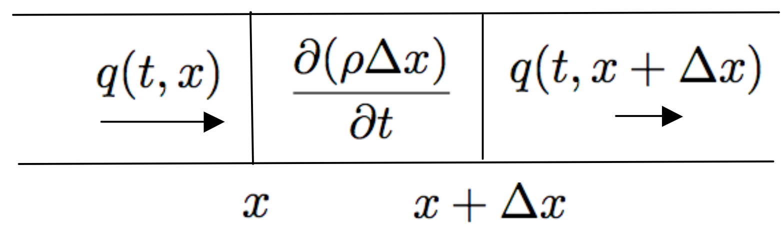
- Particles move: Conservation of mass
- Particles transmit energy (momentum): Conservation of momentum



Conservation of Mass

Recall: (see Mass Conservation and Compartmental Analysis Notes)

1-D:



$$\frac{\partial \rho}{\partial t} + \frac{\partial q}{\partial x} = 0$$

Continuity Equation

$$q = -DA\frac{\partial \rho}{\partial x} + \rho A v$$

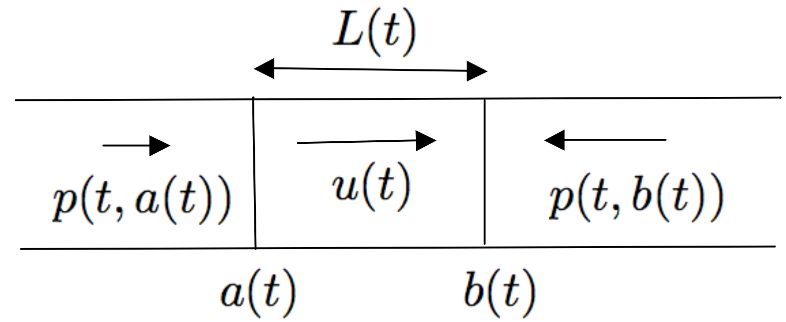
Fick's law with transport

3-D: No diffusion

$$\frac{\partial \rho}{\partial t} + \nabla \cdot (\rho \vec{u}) = 0$$

Conservation of Momentum: Lagrangian Description

Newton's Second Law: $F = \dfrac{d}{dt}(mu)$



Momentum of moving control volume at time t:

$$\int_{a(b)}^{b(t)} A\rho(t,x)u(t,x)dx$$

$$\Rightarrow \frac{d}{dt}\int_{a(t)}^{b(t)} A\rho(t,x)u(t,x)dx = Ap(t,a(t)) - Ap(t,b(t))$$

(Leibniz Rule) $$\Rightarrow \rho(t,b)u(t,b)b'(t) - \rho(t,a)u(t,a)a'(t) + \int_{a(t)}^{b(t)} \frac{\partial(\rho u)}{\partial t}dx = p(t,a) - p(t,b)$$

$$\Rightarrow \rho(t,b)u^2(t,b) - \rho(t,a)u^2(t,a) + \int_{a(t)}^{b(t)} \frac{\partial(\rho u)}{\partial t}dx = p(t,a) - p(t,b)$$

Note:

$$\rho(t,b)u^2(t,b) = \rho(t,a)u^2(t,a) + \frac{\partial(\rho u^2)}{\partial x}(t,a)L + \mathcal{O}(L^2)$$

$$p(t,b) = p(t,a) + \frac{\partial p}{\partial x}(t,a)L + \mathcal{O}(L^2)$$

$$\int_{a(t)}^{b(t)} \frac{\partial(\rho u)}{\partial t}dx = \frac{\partial(\rho u)}{\partial t}(t,a)L + \mathcal{O}(L^2)$$

Conservation of Momentum

Thus

$$\frac{\partial(\rho u^2)}{\partial x}(t,a)L + \frac{\partial(\rho u)}{\partial t}(t,a)L = -\frac{\partial p}{\partial x}(t,a)L + \mathcal{O}(L^2)$$

$$\Rightarrow \frac{\partial(\rho u^2)}{\partial x} + \frac{\partial(\rho u)}{\partial t} = -\frac{\partial p}{\partial x} \quad (*)$$

Use continuity equation (conservation of mass)

$$\frac{\partial \rho}{\partial t} + \frac{\partial(\rho u)}{\partial x} = 0$$

and the product rule to simplify $(*)$:

$$u\frac{\partial(\rho u)}{\partial x} + \rho u\frac{\partial u}{dx} + u\frac{\partial \rho}{\partial t} + \rho\frac{\partial u}{\partial t} = -\frac{\partial p}{\partial x}$$

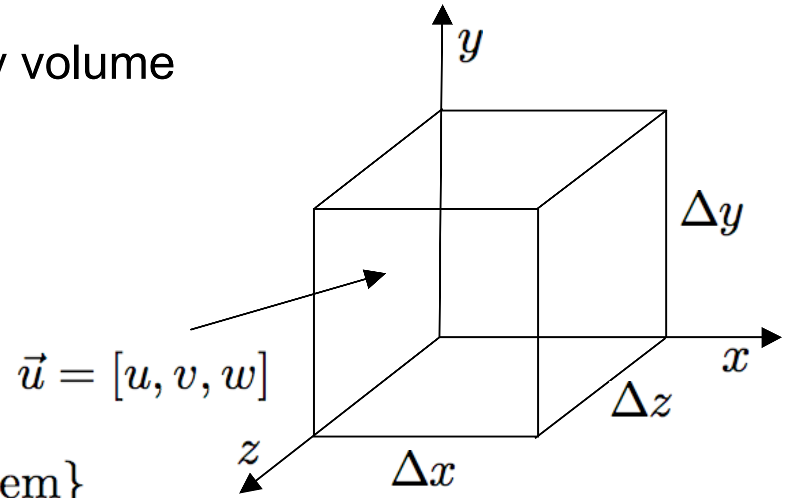
$$\Rightarrow \rho\left(\frac{\partial u}{\partial t} + u\frac{\partial u}{\partial x}\right) = -\frac{\partial p}{\partial x}$$

$$\Rightarrow \rho\frac{Du}{Dt} = -\frac{\partial p}{\partial x}$$

Euler's Equation

Conservation of Momentum: Eulerian Description

3-D Eulerian Description: Consider stationary volume



Principle:

$$\{\text{Rate of momentum accumulation}\}$$
$$= \{\text{Rate of momentum in}\}$$
$$- \{\text{Rate of momentum out}\}$$
$$+ \{\text{Sum of forces acting on system}\}$$

Momentum Change on Face x:

$$(\text{Rate of mass in}) \cdot u\big|_x = \rho u^2\big|_x \Delta y \Delta z$$

Momentum Change on Face $x + \Delta x$:

$$(\text{Rate of mass out}) \cdot u\big|_{x+\Delta x} = \rho u^2\big|_{x+\Delta x} \Delta y \Delta z$$

Conservation of Momentum: Eulerian Description

Momentum Balance: x-direction

$$\begin{aligned}
\frac{\partial(\rho u)}{\partial t}\Delta x \Delta y \Delta z &= \left[\rho u^2\big|_x - \rho u^2\big|_{x+\Delta x}\right]\Delta y \Delta z \\
&+ \left[\rho v u\big|_y - \rho v u\big|_{y+\Delta y}\right]\Delta x \Delta z \\
&+ \left[\rho w u\big|_z - \rho w u\big|_{z+\Delta z}\right]\Delta x \Delta y \\
&+ \Delta y \Delta z\left[p\big|_x - p\big|_{x+\Delta x}\right]
\end{aligned}$$

Momentum: x-component

$$\frac{\partial(\rho u)}{\partial t} = -\left[\frac{\partial}{\partial x}(\rho u^2) + \frac{\partial}{\partial y}(\rho u v) + \frac{\partial}{\partial z}(\rho u w)\right] - \frac{\partial p}{\partial x}$$

Momentum: y-component

$$\frac{\partial(\rho v)}{\partial t} = -\left[\frac{\partial}{\partial x}(\rho u v) + \frac{\partial}{\partial y}(\rho v^2) + \frac{\partial}{\partial z}(\rho v w)\right] - \frac{\partial p}{\partial y}$$

Momentum: z-component

$$\frac{\partial(\rho w)}{\partial t} = -\left[\frac{\partial}{\partial x}(\rho u w) + \frac{\partial}{\partial y}(\rho v w) + \frac{\partial}{\partial z}(\rho w^2)\right] - \frac{\partial p}{\partial z}$$

Conservation of Momentum

Note: Combination with the continuity equation yields

$$\rho \left[\frac{\partial u}{\partial t} + u \frac{\partial u}{\partial x} + v \frac{\partial u}{\partial y} + w \frac{\partial u}{\partial z} \right] = -\frac{\partial p}{dx}$$

$$\Rightarrow \rho \left[\frac{\partial u}{\partial t} + \vec{u} \cdot \nabla u \right] = -\frac{\partial p}{dx}$$

Euler's Equations:

$$\rho \left[\frac{\partial \vec{u}}{\partial t} + (\vec{u} \cdot \nabla) \vec{u} \right] = -\nabla p$$

$$\Rightarrow \rho \frac{D\vec{u}}{Dt} = -\nabla p$$

Equation of State

Conservation of Mass and Momentum:

$$\frac{\partial \rho}{\partial t} + \nabla \cdot (\rho \vec{u}) = 0$$

$$\rho \left[\frac{\partial \vec{u}}{\partial t} + (\vec{u} \cdot \nabla)\vec{u} \right] = -\nabla p$$

or

$$\frac{\partial \rho}{\partial t} + \frac{\partial (\rho u)}{\partial x} = 0$$

$$\rho \left[\frac{\partial u}{\partial t} + u \frac{\partial u}{\partial x} \right] = -\frac{\partial p}{dx}$$

Note: Need additional relation (constraint)

Barotropic Fluid: Pressure is a function only of density

$$p = f(\rho)$$

Ideal Gas: $p = \rho R T$, R is ideal gas constant

Models for Sound Waves

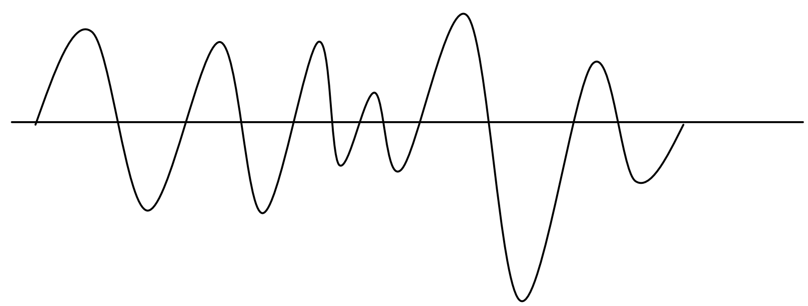
Nonlinear System:

$$\frac{\partial \rho}{\partial t} + \nabla \cdot (\rho \vec{u}) = 0$$

$$\rho \left[\frac{\partial \vec{u}}{\partial t} + (\vec{u} \cdot \nabla)\vec{u} \right] = -\nabla p$$

$$p = f(\rho)$$

Assumption: Sound is perturbations of pressure and density about static or slowly time-varying state.



Static Quantities:

$\rho_0(r)$: Static density

p_0 : Static pressure

$\vec{u}_0$: Static velocity; $\vec{u} = 0 \Rightarrow p_0$ constant in space

$\rho_0(r)$ constant in time

Models for Sound Waves

Dynamic Perturbations:

$$\rho(t,r) = \rho_0(r) + \hat{\rho}(t,r)$$

$$p(t,r) = p_0 + \hat{p}(t,r)$$

$$\vec{u}(t,r) = \hat{u}(t,r)$$

Strategy: Linearize about steady, silent case

Euler's Equation: Momentum

$$(\rho_0 + \hat{\rho})\frac{\partial \hat{u}}{\partial t} + (\rho_0 + \hat{\rho})(\hat{u}\cdot\nabla)\hat{u} = -\nabla(p_0 + \hat{p})$$

$$\Rightarrow \rho_0 \frac{\partial \hat{u}}{\partial t} + \underbrace{\hat{\rho}\frac{\partial \hat{u}}{\partial t} + (\rho_0 + \hat{\rho})(\hat{u}\cdot\nabla)\hat{u}}_{\text{H.O.T.}} = -\nabla \hat{p}$$

$$\boxed{\Rightarrow \rho_0 \frac{\partial \hat{u}}{\partial t} = -\nabla \hat{p}}$$

Models for Sound Waves

Continuity Equation: Mass

$$\frac{\partial}{\partial t}(\rho_0 + \hat{\rho}) + \nabla \cdot ((\rho_0 + \hat{\rho})\hat{u}) = 0$$

$$\Rightarrow \frac{\partial \hat{\rho}}{\partial t} + \nabla \cdot (\rho_0 \hat{u}) + \underline{\nabla \cdot (\hat{\rho}\hat{u})} = 0$$

$$\boxed{\Rightarrow \frac{\partial \hat{\rho}}{\partial t} + \nabla \cdot (\rho_0 \hat{u}) = 0}$$

Equation of State:

$$\begin{aligned} p &= f(\rho) \\ &= f(\rho_0 + \hat{\rho}) \\ &= f(\rho_0) + f'(\rho_0)\hat{\rho} + \text{H.O.T.} \\ &= p_0 + f'(\rho_0)\hat{\rho} + \text{H.O.T.} \end{aligned}$$

First-Order Relation:

$\boxed{\hat{p} = c^2 \hat{\rho}}$ where $c^2 \equiv f'(\rho_0) = \frac{\partial p}{\partial \rho}\Big|_{\rho_0}$ is the speed of sound in the material

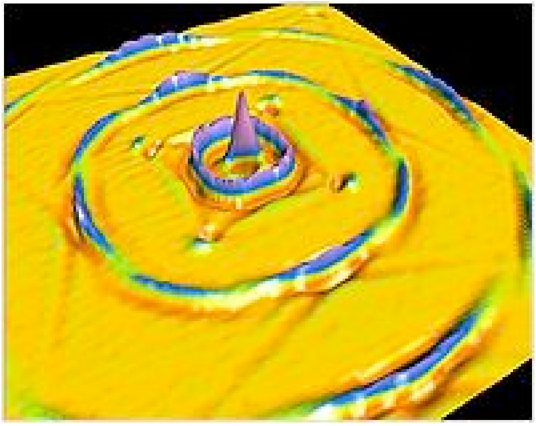
Wave Equation in Pressure

First-Order Equations for Sound:

$$\rho_0 \frac{\partial \hat{u}}{\partial t} = -\nabla \hat{p} \quad \text{(Euler)}$$

$$\frac{\partial \hat{\rho}}{\partial t} + \nabla \cdot (\rho_0 \hat{u}) = 0 \quad \text{(Continuity)}$$

$$\hat{p} = c^2 \hat{\rho} \quad \text{(State)}$$



Wave Equation in Pressure:

$$\frac{\partial}{\partial t}(\hat{p}/c^2) = -\nabla \cdot (\rho_0 \hat{u}) \qquad \text{(State into Continuity)}$$

$$\Rightarrow \frac{1}{c^2}\frac{\partial^2 \hat{p}}{\partial t^2} = -\nabla \cdot \left(\rho_0 \frac{\partial \hat{u}}{\partial t}\right) \qquad \text{(Differentiate wrt time)}$$

$$\Rightarrow \frac{1}{c^2}\frac{\partial^2 \hat{p}}{\partial t^2} = -\nabla \cdot (-\nabla \hat{p}) \qquad \text{(Substitute Euler)}$$

$$\boxed{\Rightarrow \frac{\partial^2 \hat{p}}{\partial t^2} = c^2 \Delta \hat{p}}$$

Advantages: 1-D equation, no additional assumptions

Disadvantages: $\hat{p}$ may or may not be easy to measure, may not facilitate multiphysics coupling

Wave Equation in Velocity

Wave Equation in Velocity: Note that

$$\rho_0 \frac{\partial^2 \hat{u}}{\partial t^2} = -\nabla \hat{p}_t \qquad \text{(Differentiate Euler)}$$

and

$$\begin{aligned} \hat{p}_t &= c^2 \hat{\rho}_t \qquad \text{(Differentiate state)} \\ &= -c^2 \nabla \cdot (\rho_0 \hat{u}) \qquad \text{(Continuity)} \end{aligned}$$

Thus

$$\rho_0 \frac{\partial^2 \hat{u}}{\partial t^2} = \nabla \left[c^2 \nabla \cdot (\rho_0 \hat{u}) \right]$$

Note: Holds for variable $\rho_0(r)$

Additional Assumption: ρ_0 constant

$$\frac{\partial^2 \hat{u}}{\partial t^2} = c^2 \Delta \hat{u} + c^2 \nabla \times (\nabla \times \hat{u})$$

Vector Identity

$$\nabla \times (\nabla \times \vec{w}) = \nabla [\nabla \cdot \vec{w}] - \Delta \vec{w}$$

Irrotational Flow:

$$\frac{\partial^2 \hat{u}}{\partial t^2} = c^2 \Delta \hat{u}$$

Advantage: May be able to directly measure velocity

Disadvantage: Requires 3 variables,
added assumption of constant ρ_0

Wave Equation in Potential

Initial Assumption: ρ_0 constant

The curl of Euler's equation yields

$$\nabla \times \left(\rho_0 \frac{\partial \hat{u}}{\partial t}\right) = -\nabla \times (\nabla \hat{p})$$

$$\Rightarrow \frac{\partial}{\partial t}(\nabla \times \hat{u}) = 0 \qquad \text{since } \nabla \times (\nabla f) = 0\,,\ \rho_0 \text{ constant}$$

Initial Vorticity: Suppose irrotational

$$\nabla \times \hat{u}\big|_{t=0} = 0$$

$$\Rightarrow \nabla \times \hat{u} = 0 \ \text{ for all time}$$

$$\Rightarrow \hat{u} \text{ is a conservative field}$$

$$\Rightarrow \text{There exists a nonunique scalar potential } \phi \text{ such that } \hat{u} = -\nabla \phi$$

Thus

$$\rho_0 \nabla \phi_t = \nabla \hat{p} \qquad \text{(Euler's equation)}$$

$$\Rightarrow \nabla(\rho_0 \phi_t - \hat{p}) = 0$$

$$\Rightarrow \rho_0 \phi_t - \hat{p} = -k(t)$$

Wave Equation in Potential

WLOG: Take $k \equiv 0$. To see why, define new potential

$$\tilde{\phi} = \phi + \frac{1}{\rho_0} \int_0^t k(s) ds$$

$$\Rightarrow \nabla \tilde{\phi} = \nabla \phi = -\vec{u}$$

and

$$\rho_0 \tilde{\phi}_t = \rho_0 \phi_t + k(t) = \hat{p}$$

WLOG: Take $\hat{p} = \rho_0 \phi_t$. Thus

$$\frac{\partial \hat{p}}{\partial t} = -\nabla \cdot (\rho_0 \hat{u}) \qquad \text{(Continuity equation)}$$

$$\Rightarrow \frac{\partial}{\partial t}\left(\frac{\hat{p}}{c^2}\right) = \nabla(\rho_0 \nabla \phi) \qquad \text{(State equation)}$$

$$\Rightarrow \frac{\partial}{\partial t}\left(\frac{\rho_0 \phi_t}{c^2}\right) = \rho_0 \Delta \phi$$

$$\boxed{\Rightarrow \phi_{tt} = c^2 \Delta \phi}$$

Advantages: Relations $\hat{u} = -\nabla \phi$ and $\hat{p} = \rho_0 \phi_t$ can facilitate coupling with structural systems

Disadvantage: ϕ is difficult to measure directly

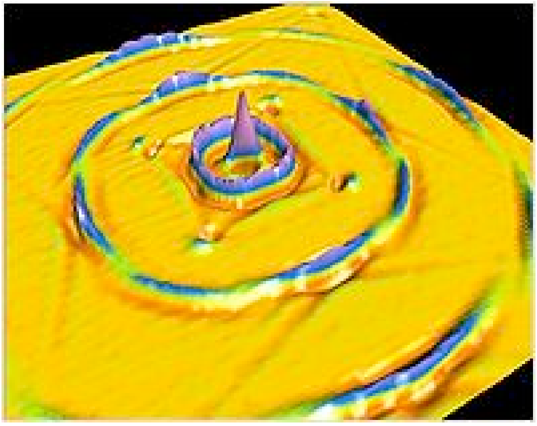
Wave Equation in Potential

Nonconstant Density: ρ_0 not constant in space

Wave Equation in Potential: See book, pages 236-237

$$\Phi_{tt} = c^2 \Delta \Phi$$

Linear Wave Equations for Sound



Pressure:

$$\frac{\partial^2 \hat{p}}{\partial t^2} = c^2 \Delta \hat{p}$$

Boundary Conditions

Initial Conditions

Velocity:

$$\frac{\partial^2 \hat{u}}{\partial t^2} = c^2 \Delta \hat{u}$$

Boundary Conditions

Initial Conditions

Potential:

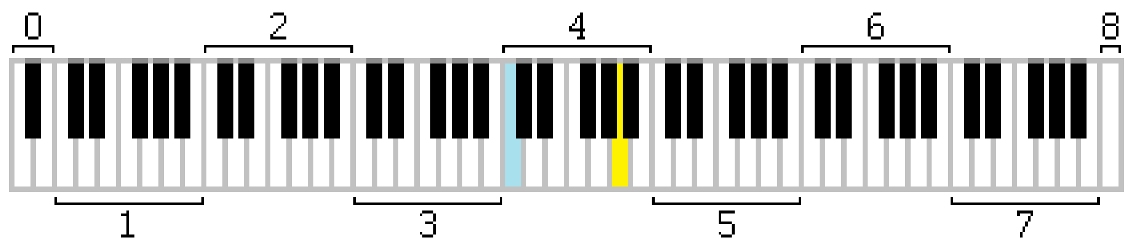
$$\frac{\partial^2 \phi}{\partial t^2} = c^2 \Delta \phi$$

Boundary Conditions

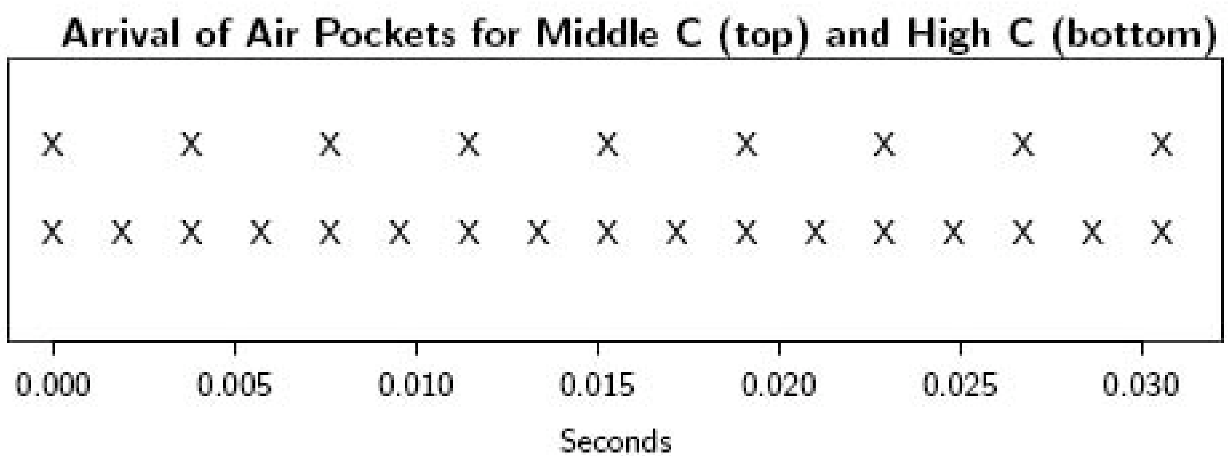
Initial Conditions

Music

Pitch: This is simply the frequency of pure tones; e.g., Middle C is about 261.6 Hz as calculated in relation to A above Middle C which as a frequency of 440 Hz.

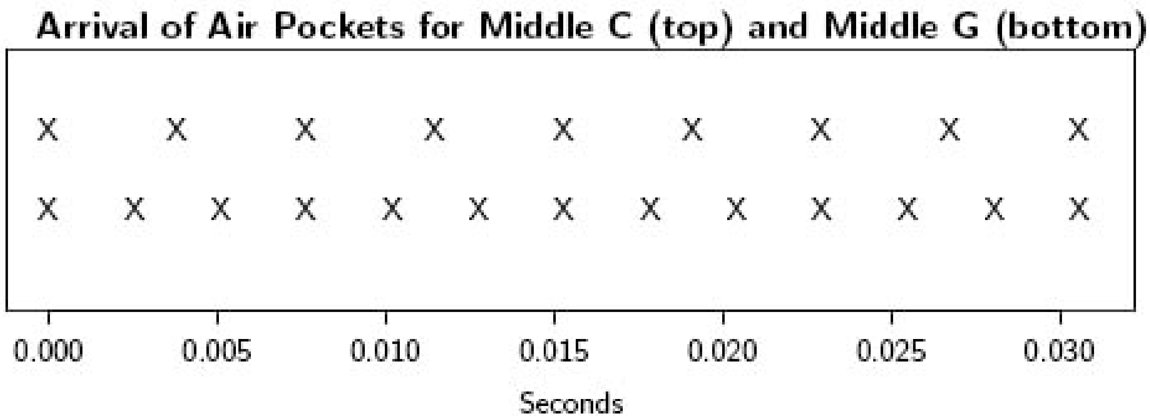


Note: Going up an octave doubles the frequency. This is easily observed on a guitar where pressing at the twelfth fret cuts the string in half which doubles the frequency and raises the tone an octave.



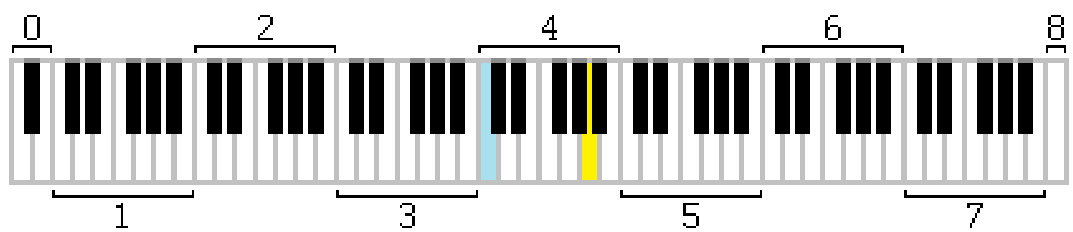
Music

Note: Going up a fifth (e.g., C and G) causes frequencies to align every third interval for the G --- e.g., First two notes of *Star Wars* theme.



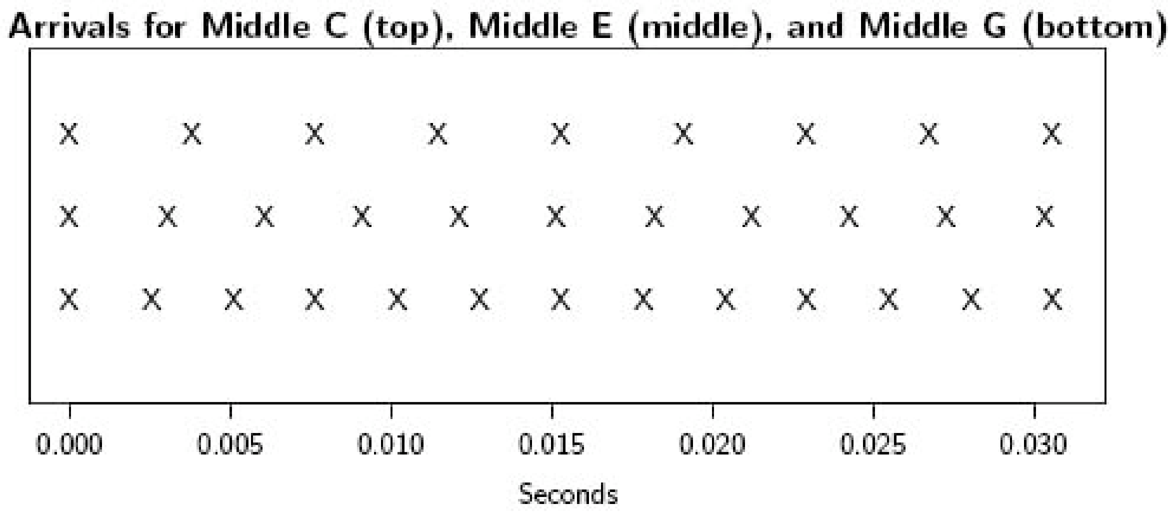
Reason: Equal tempering which spaces all 12 notes equally

- Each semitone is thus $\sqrt[12]{2} \approx 1.059463$
- C sharp thus has a frequency of $261.6 \cdot 1.059463$
- G has a frequency of $261.6 \cdot (1.059463^7) = 261.6 \cdot 1.498307 \approx 261.6 \cdot (3/2)$
- A: $261.6 \cdot (1.059463^9) = 440$



Music

Note: Include a 3rd (E) and a fifth (G) -- C major chord



Note: Raising the key causes the confluence to occur more quickly.

Music

Note: The wavelengths of notes is related to their frequency by the expression

$$\lambda = \frac{c}{f}$$

where c = 343 m/s for air.

Example: The wavelength of Middle C is thus 1.31 m.