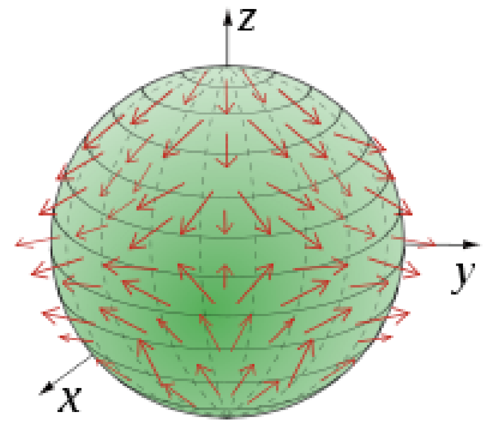
Aspects of Vector Calculus

"O could I flow like thee, and make thy stream
My great example, as it is my theme!
Though deep, yet clear, though gentle, yet not dull,
Strong without rage, without o'erflowing full"

Sir John Denham

Vector and Gradient Fields

Vector Fields: A vector field is a function that assigns a vector to each point in its domain.

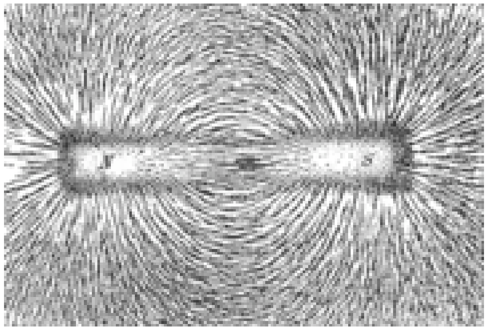


Examples:

- Wind speeds and directions on the surface of the earth
- Velocity of a moving fluid
- Electromagnetic fields
- Gravitational fields

Gradient Fields: The gradient field of a differentiable function f(x,y,z) is the field of gradient vectors

$$\nabla f = \frac{\partial f}{\partial x}\hat{\imath} + \frac{\partial f}{\partial y}\hat{\jmath} + \frac{\partial f}{\partial z}\hat{k}$$

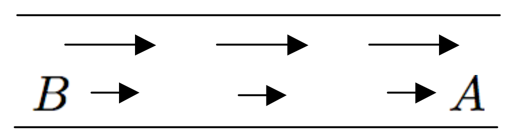


Conservative Fields and Potentials

Definition: Let $\vec{F}$ be a field defined on a domain D and suppose that for any two points A and B in D, the work $\int_A^B \vec{F} \cdot d\vec{r}$ is the same over all paths from A to B. Then the field $\vec{F}$ is conservative on D.

Examples:

- Gravitational fields
- Flow fields



Definition: If $\vec{F}$ is a field and $\vec{F} = \nabla f$ for some scalar function f, then f is a potential function for F.

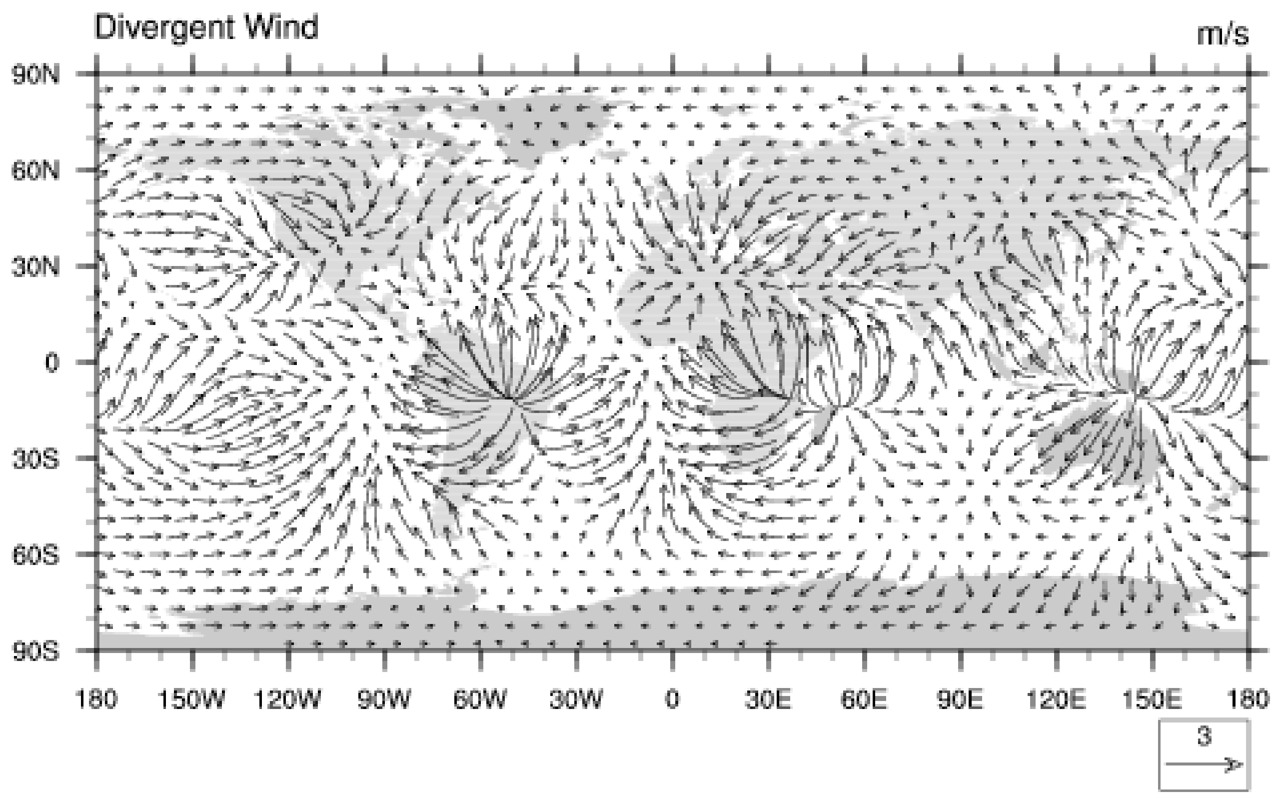
Example: Gravitational potential energy

$$U(z) = \int_0^z mg\,dz = mgz$$

Divergence

Definition: The divergence (flux density) of a vector field is an operator that quantifies the magnitude of a vector field's source or sink at a point (x,y,z). Mathematically, it is given by

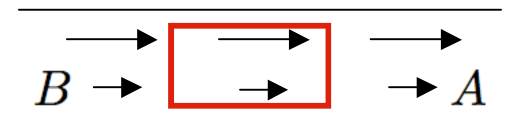
$$\mathrm{div}\vec{F} = \nabla \cdot \vec{F}$$



Curl

Definition: If $\vec{r}$ is a smooth curve in the domain of a continuous velocity field $\vec{F}$, the flow along the curve from a to b is

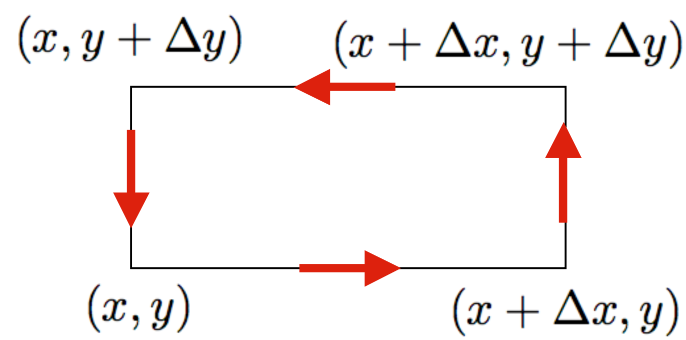
$$\text{Flow} = \int_a^b \vec{F} \cdot \hat{T} ds.$$



If the curve is closed, the flow is termed the circulation around the curve.

Spin in 2-D: Consider the velocity field

$$\vec{F}(x,y) = M(x,y)\hat{\imath} + N(x,y)\hat{\jmath}$$



Approximate Flows:

Top: $\vec{F}(x, y+\Delta y) \cdot (-\hat{\imath})\Delta x = -M(x, y+\Delta y)\Delta x$

Bottom: $\vec{F}(x,y) \cdot \hat{\imath}\Delta x = M(x,y)\Delta x$

Right: $\vec{F}(x+\Delta x, y) \cdot \hat{\jmath}\Delta y = N(x+\Delta x, y)\Delta y$

Left: $\vec{F}(x,y) \cdot (-\hat{\jmath})\Delta y = -N(x,y)\Delta y$

Curl

Counter-clockwise Circulation (Right hand rule): Sum of flows

Top and bottom:

$$-[M(x, y + \Delta y) - M(x, y)]\Delta x \approx -\left(\frac{\partial M}{\partial y}\Delta y\right)\Delta x$$

Right and left:

$$[N(x + \Delta x, y) - N(x, y)]\Delta y \approx \left(\frac{\partial N}{\partial x}\Delta x\right)\Delta y$$

Circulation Density:

$$\frac{\text{Circulation around rectangle}}{\text{Rectangle Area}} \approx \frac{\partial N}{\partial x} - \frac{\partial M}{\partial y}$$

K-Component of the Curl:

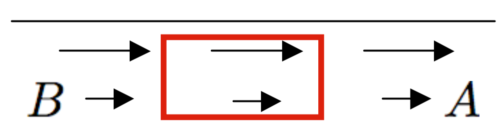
$$(\text{curl } \vec{F}) \cdot \vec{k} = \frac{\partial N}{\partial x} - \frac{\partial M}{\partial y}$$

Note: Irrotational if

$$\nabla \times \vec{F} = 0$$

Example: $\vec{F}(x, y) = y\hat{\imath}$

$$\nabla \times \vec{F} = -\hat{k}$$



Green's Theorem

Flux-Divergence Form: Let C be a piecewise smooth, simple closed curve enclosing a region R in the plane and let $\vec{F} = M\hat{\imath} + N\hat{\jmath}$. Then

$$\oint_C \vec{F} \cdot \hat{n} ds = \oint_C M dy - N dx = \iint_R \left(\frac{\partial M}{\partial x} + \frac{\partial N}{\partial y} \right) dx dy$$

Flux in/out of region

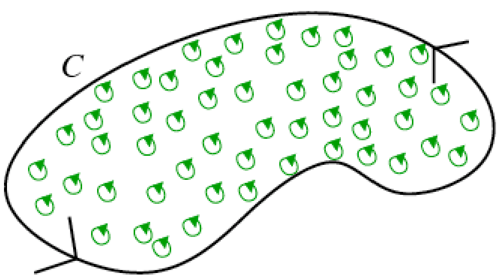
Flux in/out of "micro"-region

Circulation-Curl Form:

$$\oint_C \vec{F} \cdot \hat{T} ds = \oint_C M dx + N dy = \iint_R \left(\frac{\partial N}{\partial x} - \frac{\partial M}{\partial y} \right) dx dy$$

"Macrocirculation" of F

"Microcirculation" of F



Equivalence:

- Application of first to $\vec{G}_1 = N\hat{\imath} - M\hat{\jmath}$ yields second
- Application of second to $\vec{G}_2 = -N\hat{\imath} + M\hat{\jmath}$ yields first

Curl

3-D Representation:

$$\nabla \times \vec{F} = \begin{vmatrix} \hat{\imath} & \hat{\jmath} & \hat{k} \\ \frac{\partial}{\partial x} & \frac{\partial}{\partial y} & \frac{\partial}{\partial z} \\ M & N & P \end{vmatrix}$$

$$= \left(\frac{\partial P}{\partial y} - \frac{\partial N}{\partial z}\right)\hat{\imath} + \left(\frac{\partial M}{\partial z} - \frac{\partial P}{\partial x}\right)\hat{\jmath} + \left(\frac{\partial N}{\partial x} - \frac{\partial M}{\partial y}\right)\hat{k}$$

Easy Result:

$$\nabla \times \nabla f = \begin{vmatrix} \hat{\imath} & \hat{\jmath} & \hat{k} \\ \frac{\partial}{\partial x} & \frac{\partial}{\partial y} & \frac{\partial}{\partial z} \\ \frac{\partial f}{\partial x} & \frac{\partial f}{\partial y} & \frac{\partial f}{\partial z} \end{vmatrix} = (f_{zy} - f_{yz})\,\hat{\imath} + (f_{zx} - f_{xz})\,\hat{\jmath} + (f_{yx} - f_{xy})\,\hat{k} = 0$$

Equivalent Statements:

- $\vec{F}$ conservative on D
- $\vec{F} = \nabla f$ on D
- $\oint_C \vec{F} \cdot d\hat{r} = 0$ over any closed path in D
- $\nabla \times \vec{F} = 0$ throughout D

"Big" Theorems

Normal form of Green's Theorem: $$\oint_C \vec{F} \cdot \hat{n} ds = \iint_R \nabla \cdot \vec{F} dA$$

Divergence Theorem: $$\iint_S \vec{F} \cdot \hat{n} d\sigma = \iiint_D \nabla \cdot \vec{F} dV$$

Tangential form of Green's Theorem: $$\oint_C \vec{F} \cdot d\hat{r} = \iint_R \nabla \times \vec{F} \cdot \hat{k} dA$$

Stoke's Theorem: $$\oint_C \vec{F} \cdot d\hat{r} = \iint_S \nabla \times \vec{F} \cdot \hat{n} d\sigma$$

S can change as long a C does not!

Fundamental Theorem of Calculus: $$F(b) - F(a) = \int_a^b \frac{dF}{dx} dx$$

Other Important Results

Note: See Appendix B of Text