Model:

$$\boxed{S} \longrightarrow \boxed{I} \longrightarrow \boxed{R}$$

Simplest Model 1: Kermack & McKendrick [1927] No births/deaths

$$\frac{dS}{dt} = -\beta \frac{I}{N} S \quad , \quad S(0) = S_0$$

$$\frac{dI}{dt} = \beta \frac{I}{N} S - \gamma I \quad , \quad I(0) = I_0 \qquad (*)$$

$$\frac{dR}{dt} = \gamma I \quad , \quad R(0) = R_0$$

Note: $\frac{dS}{dt} + \frac{dI}{dt} + \frac{dR}{dt} = 0$

$\Rightarrow S(t) + I(t) + R(t) = N$ Total Population Size

And

$\frac{I}{N}$: Fraction infected $\left(\frac{\#}{\#}\right)$ γ: Recovery rate

β: Infection rate $\left(\frac{\circ}{\text{day}}\right)$

$\Rightarrow \left(\frac{\#}{\text{day}}\right) = \left(\frac{\circ}{\text{Day}}\right)(\#)$

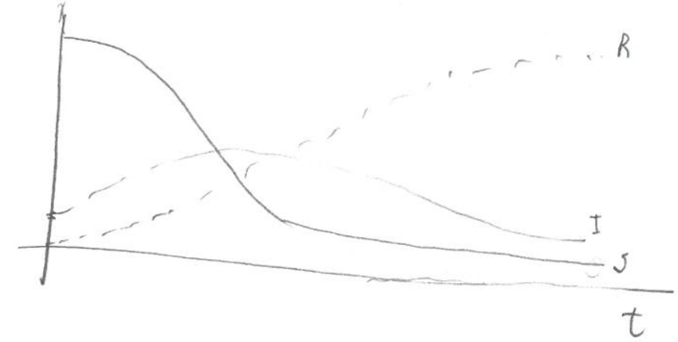
p30: Rate at which infections occur is βI or β per infectious individual

Note: Some others use

$$\frac{dS}{dt} = -\beta I S \qquad \text{(Mass action)} \; ; \; \beta \left(\frac{\circ}{\#\ \text{day}}\right)$$

(*) Considered to be better : Note Normalize to 1

Typical Plot:



Model 2:

$$\frac{dS}{dt} = -K\nu \frac{I}{N} S$$

$$\frac{dI}{dt} = K\nu \frac{I}{N} S - \gamma I$$

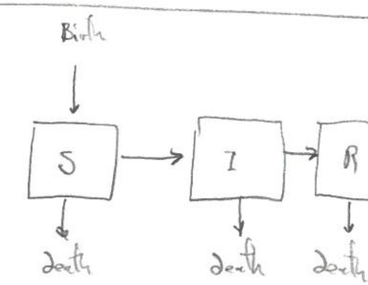
$$\frac{dR}{dt} = \gamma I$$

K: contact rate $\left(\frac{\#}{\text{day}}\right)$

ν: Infection coef $\left(\frac{\circ}{\#}\right)$

Reason: Quarantine

Model 3: μ = Birth & death rate



$$\frac{dS}{dt} = \mu N - \mu S - \beta \frac{I}{N} S$$

$$\frac{dI}{dt} = \beta \frac{I}{N} S - (\gamma + \mu) I$$

$$\frac{dR}{dt} = \gamma I - \mu R$$

p 29 of Lloyd

Note: $\frac{dS}{dt} + \frac{dI}{dt} + \frac{dR}{dt} = 0$ so $S(t) + I(t) + R(t) = N$

Model Behavior: Consider first

$$\frac{dI}{dt} = I\left[\frac{\beta S}{N} - \gamma\right] \qquad (+)$$

i) $\frac{\beta S}{N} < \gamma$, $\frac{dI}{dt} < 0$

ii) $\frac{\beta S}{N} > \gamma$, $\frac{dI}{dt} > 0$

$$\Rightarrow \frac{dI}{dt} = 0 \Rightarrow S^* = \frac{N}{R_0}$$

Initial Epidemic Behavior: Initially $S(t) \simeq N$ so initial growth rate r

$$r = \beta - \gamma$$
$$= \gamma(R_0 - 1) \qquad \text{where } R_0 \equiv \frac{\beta}{\gamma}$$

(Basic reproductive number of pop)

Note: Infection spreads only if $R_0 > 1$

Biological Interpretation: Average number of secondary infections caused by introduction of single infectious individual into susceptible population.

Note: $S(t) \simeq N$ Linearizes problem

Epidemic Progresses: $R_0 > 1$ Eventually $S(t) < \frac{N}{R_0}$ and infection slows

Define: Average number of secondary infections at time t: R_t

$$R_t = R_0\left(\frac{S(t)}{N}\right)$$

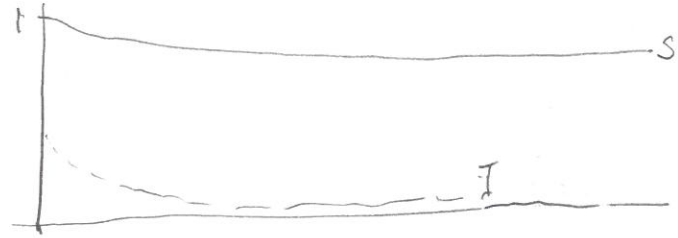
Decreases linearly as S decreases

Then (+) is

$$\frac{dI}{dt} = I\left[\frac{\beta S}{N} - \gamma\right]$$
$$= \gamma I\left[\frac{\beta S}{\gamma N} - 1\right]$$
$$= \gamma I\left(R_t - 1\right)$$

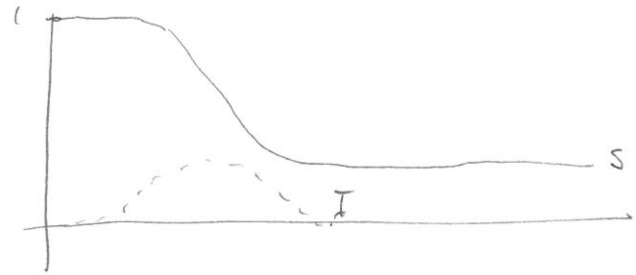
$R_t < 1 \Rightarrow \frac{dI}{dt} < 0$

$R_t > 1 \Rightarrow \frac{dI}{dt} > 0$

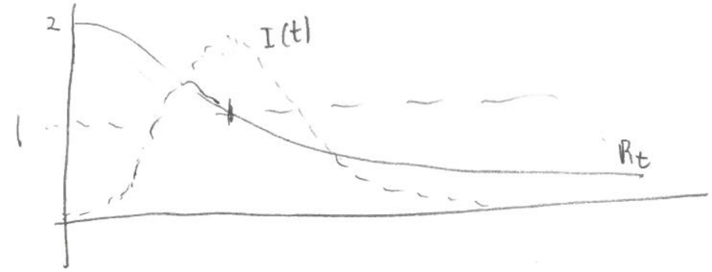


No Epidemic

$R_0 < 1$



Self-Limiting Epi:



Size of Epidemic: **Strategy:** Remove time – Recall that

$S \longrightarrow I \longrightarrow t$

$$\frac{dS}{dt} = \frac{dS}{dI}\frac{dI}{dt} \Rightarrow \frac{dS}{dI} = \frac{\left(\frac{dS}{dt}\right)}{\left(\frac{dI}{dt}\right)}$$

Now

$$\frac{\dot{S}}{\dot{I}} = \frac{-\beta SI/N}{\beta SI/N - \gamma I}$$
$$= \frac{-1}{1 - \gamma N/(\beta S)}$$

$$\Rightarrow \frac{dS}{dI} = \frac{-1}{1 - N/(R_0 S)}$$

Thus:

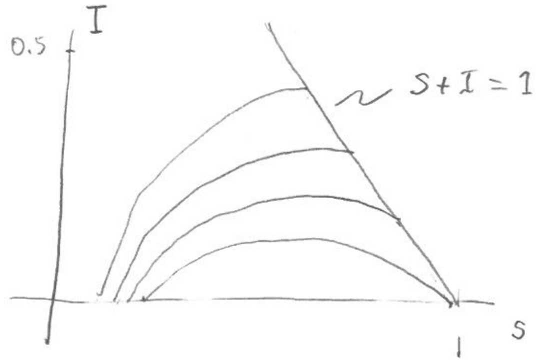
$$\left(1 - \frac{N}{R_0 S}\right)\frac{dS}{dI} = -1$$

$$\Rightarrow \int \left(1 - \frac{N}{R_0 S}\right) dS = \int -dI$$

$$\Rightarrow S - \frac{N}{R_0}\ln S = -I + C$$

Result: $\underbrace{S - \frac{N}{R_0}\ln S + I = C}$

Constant along curves - Conserved quantity : Goal: find size of epidemic



- Compare conserved quantity at $t=0$ and $t=\infty$.

$t=0$ $\quad S \simeq N,\ I \simeq 0$

$t=\infty$ $\quad I(t) \simeq 0$

- Equate $N - \frac{N}{R_0}\ln N = S(\infty) - \frac{N}{R_0}\ln S(\infty)$

$$\Rightarrow N - S(\infty) = \frac{N}{R_0}\ln\frac{N}{S(\infty)} \qquad \left(-\frac{R_0}{N}\right)$$

$$\Rightarrow -R_0\left(1 - \frac{S(\infty)}{N}\right) = \ln\frac{S(\infty)}{N} \qquad -\ln x = \ln\frac{1}{x}$$

$$\Rightarrow \exp\left(-R_0\left[1 - \frac{S(\infty)}{N}\right]\right) = \underbrace{\frac{S(\infty)}{N}}_{\text{Fraction who escaped infection}}$$

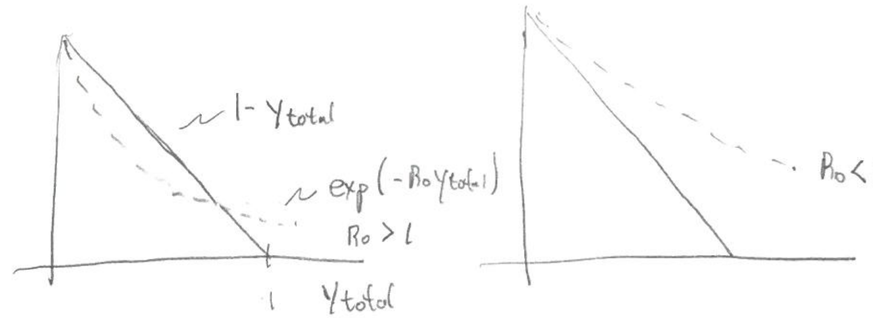
Note: End of epidemic: $I(t) = 0$

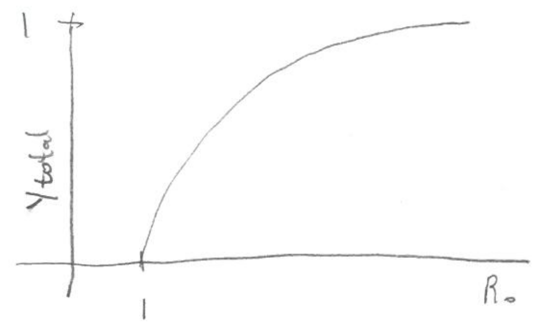
Note: $Y_{total} = 1 - \frac{S(\infty)}{N}$ is epidemic size

Thus

$$\boxed{\exp(-R_0 Y_{total}) = 1 - Y_{total}}$$

Transcendental eq





Recall: $S - \frac{N}{R_0} \ln S + I = C$ Independent of Time

Assumption 1: $I(0) \approx 0$, $S(0) \approx N$, $R(0) = 0$

$I(\infty) \approx 0$

Also,

$$\frac{dS}{dR} = \frac{dS}{dt} \Big/ \frac{dR}{dt} = \frac{-\frac{\beta I}{N} S}{\gamma I} = -\frac{\beta}{\gamma} \frac{S}{N}$$

$$\Rightarrow \frac{dS}{dR} = -\frac{R_0}{N} S$$

$$\Rightarrow \int \frac{1}{S} dS = -\frac{R_0}{N} \int dR$$

$$\Rightarrow -\frac{R_0}{N} R = \ln(S) + K_1$$

$$\Rightarrow S = K_1 e^{-\frac{R_0}{N} R}$$

$$\Rightarrow S(t) = S(0) e^{-\frac{R_0}{N} R(t)}$$

$$\Rightarrow S(\infty) \approx N e^{-\frac{R_0}{N} R(\infty)}$$

$$\Rightarrow N - R(\infty) \approx N e^{-\frac{R_0}{N} R(\infty)}$$

$$\Rightarrow R(\infty) \approx N\left(1 - e^{-\frac{R_0}{N} R(\infty)}\right)$$

Observation: 1) Not all Recover!

2) Not all get sick!!

Numerical Solution: ode45.m or ode15s.m

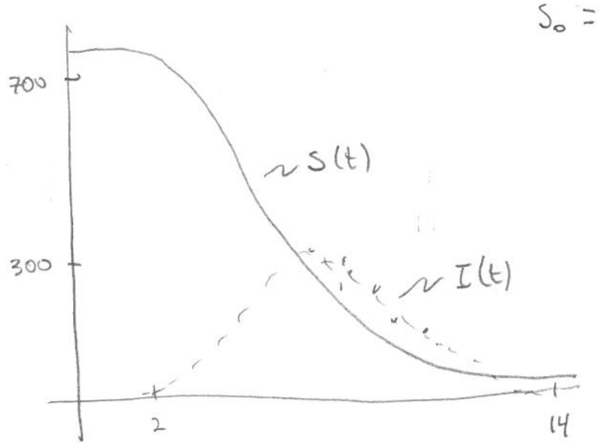
:

Example 1: Influenza outbreak, The Lancet 1978 — Project!!

British boarding school

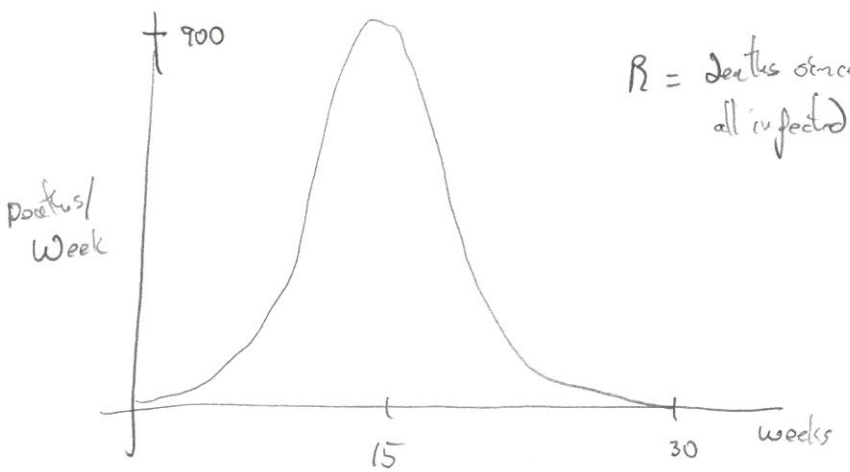
$N = 763$

$S_0 = 762$



Example 2: Kermack & McKendrick — Plague in Bombay

R = Deaths since almost all infected died



Control of Infection: **Goal:** Reduce R_0 so $R_0 < 1$

Recall: $R_0 = \beta \cdot$ average duration of infectiousness

Both infectiousness and rate of contacts

Thus

- Shorten infectious period
- Reduce infectiousness of sick folks

— Vaccine

- Reduce number of contacts

Note: Model 3

$$\frac{dI}{dt} = \beta \frac{I}{N} S - (\gamma + \mu) I$$

$$= I \left[\frac{\beta}{\gamma+\mu} \frac{S(t)}{N} - 1 \right] (\gamma+\mu)$$

$$= I \left[\underbrace{R_0 \frac{S(t)}{N} - 1}_{R_t - 1} \right] (\gamma+\mu)$$

Average Duration of infectiousness: $\tau = \frac{1}{\gamma+\mu}$

$$R_0 = \frac{\beta}{\gamma+\mu} = \beta\tau$$

Entirely susceptible population

More generally, consider

$$R_t = R_0 \left(\frac{S(t)}{N} \right)$$

Goal: Maintain $S(t) < \frac{N}{R_0}$

Note: Do not need $S(t) = 0$!

Goal: Vaccinate at least the critical percentage

$$p_c = 1 - \frac{1}{R_0}$$

Herd Immunity

which leaves $\frac{1}{R_0}$ of population in susceptible class.

Note:

	R_0	p_c
smallpox	4-5	80%
polio	5-7	86%
chickenpox	8-10	90%
measles	15	93%

Anderson & May

Vaccination at Birth: Fraction p

$$\frac{dS}{dt} = \mu(1-p)N - \beta SI/N - \mu S$$

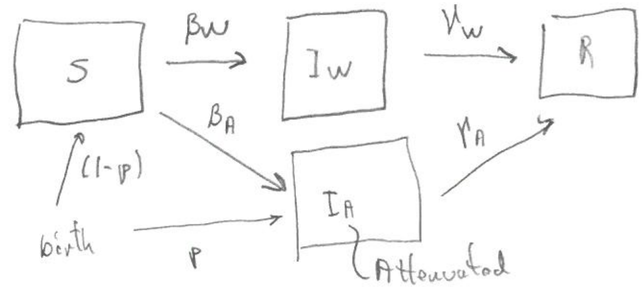
$$\frac{dI}{dt} = \frac{\beta SI}{N} - (\gamma + \mu)I \qquad \text{Unchanged!}$$

$$\frac{dR}{dt} = \gamma I - \mu R + \mu p N$$

Note: $\frac{dI}{dt} = 0 \Rightarrow S^* = \frac{N}{R_0}$ Hence same number of susceptibles at equilibrium

Result: Get infected older!

Alternative Vaccination Model: e.g, Polio



Model:

$$\frac{dS}{dt} = \mu(1-p)N - \beta_w S I_w/N - \beta_A S I_A/N - \mu S$$

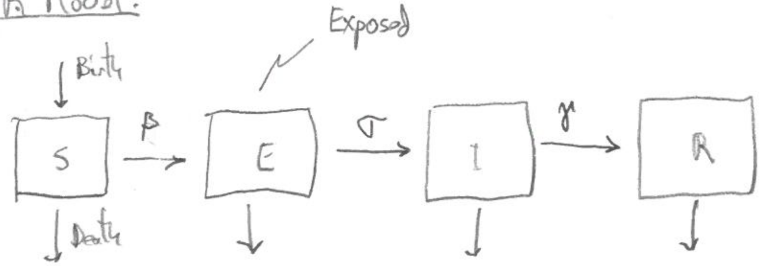
$$\frac{dI_w}{dt} = \beta_w S I_w/N - (\gamma_w + \mu) I_w$$

$$\frac{dI_A}{dt} = \beta_A S I_A/N - (\gamma_A + \mu) I_A + \mu N p$$

Take

$$R_0^w = \frac{\beta_w}{\gamma_w + \mu}, \quad R_0^A = \frac{\beta_A}{\gamma_A + \mu}$$

SEIR Model:



$$\frac{dS}{dt} = \mu N - \beta \frac{SI}{N} - \mu S$$

$$\frac{dE}{dt} = \frac{\beta S I}{N} - (\sigma + \mu) E$$

Note: Introduces exponential latency period

$$\frac{dI}{dt} = \sigma E - (\gamma + \mu) I$$

$$\frac{dR}{dt} = \gamma I - \mu R$$

γ, σ large compared to μ

Note: Average duration of infectious: $\tau = \dfrac{1}{\gamma + \mu} \simeq \dfrac{1}{\gamma}$

" latency: $\tau' = \dfrac{1}{\sigma + \mu} \simeq \dfrac{1}{\sigma}$

Probability of becoming infectious: $P = \dfrac{\sigma}{\sigma + \mu}$

$$= \frac{1}{1 + \mu/\sigma}, \quad \frac{\mu}{\sigma} \text{ small}$$

$$\simeq 1 - \frac{\mu}{\sigma}$$

Close to 1

Note: $(1+x)^n \simeq 1 + nx$ for small x

Now,

Av rate

Duration

$$R_0 = \beta \tau P$$

$$= \frac{\beta}{\gamma + \mu} \cdot \frac{\sigma}{\sigma + \mu}$$

Note: As before, $R_0 > 1 \Rightarrow$ Infection

Note: ** Exposed class: Introduces delay between acquisition of infection and transmission of infection

Linearization: Take $S \simeq N$

$$\frac{dE}{dt} = \beta I - (\sigma + \mu) E$$

$$\frac{dI}{dt} = \sigma E - (\gamma + \mu) I$$

Growth Rate: Determined by eigenvalues of Jacobian

$$J = \begin{bmatrix} -(\sigma+\mu) & \beta \\ \sigma & -(\gamma+\mu) \end{bmatrix}$$

Eigenvalues: r satisfy $\det(J - rI) = 0$

$$0 = (\mu+\sigma+r)(\mu+\gamma+r) - \sigma\beta$$

$$= r^2 + (\mu+\sigma+\mu+\gamma)r + (\mu+\sigma)(\mu+\gamma) - \sigma\beta$$

$$= r^2 + \left(\frac{1}{\tau'} + \frac{1}{\tau}\right)r + (\mu+\sigma)(\mu+\gamma)\left[1 - \frac{\sigma\beta}{(\mu+\sigma)(\mu+\gamma)}\right]$$

$$= r^2 + \left(\frac{1}{\tau'} + \frac{1}{\tau}\right)r + \frac{1}{\tau'\tau}\underbrace{(1 - R_0)}_{\text{Note}}$$

Special Case: $R_0 \gtrsim 1$ but close to 1 $\Rightarrow r$ small

$\Rightarrow r^2$ very small

Then

$$\left(\frac{1}{\tau'} + \frac{1}{\tau}\right)r + \frac{1}{\tau'\tau}(1 - R_0) \simeq 0$$

$$\Rightarrow \boxed{r \approx \frac{1}{\tau+\tau'}(R_0 - 1)}$$

Slows rate at which epidemic spreads

$$\boxed{\text{SIR: } r = \frac{1}{\tau}(R_0 - 1)}$$

Note: $R_0 > 1$

$$S^* = \frac{N}{R_0}$$

$$E^* = \mu N\left(1 - \frac{1}{R_0}\right)\frac{1}{\mu+\sigma}$$

$$I^* = \frac{\mu N}{\mu+\gamma}\left(1 - \frac{1}{R_0}\right)\frac{\sigma}{\mu+\sigma}$$

Jacobian: Probably not worth it...

Note: SIR and SEIR are relatively good but miss mechanisms

Stochasticity: Number of random events depends on population size!

- E.g., Infected to contact anyone

Deterministic: Treat numbers in each class as continually varying.

e.g.)

$$\frac{dS}{dt} = \mu N - \mu S - \frac{\beta S I}{N}$$

$$\frac{dI}{dt} = \frac{\beta S I}{N} - \mu I - \mu I$$

Event	Transition	Rate at Which Event Occurs	Probability of transition in interval $[t, t+dt]$
Birth	$S \to S+1$	μN	$\mu N\,dt$
S. Death	$S \to S-1$	μS	$\mu S\,dt$
Infection	$S \to S-1,\ I \to I+1$	$\frac{\beta S I}{N}$	$\left(\frac{\beta S I}{N}\right)dt$
Recovery	$I \to I-1$	γI	$\gamma I\,dt$
Inf. Death	$I \to I-1$	μI	$\mu I\,dt$

Stochastic Formulation: Assume that population composed of individuals and transitions are discrete events.

- Deterministic model gives average rate

Poisson Distribution: Discrete distribution that expresses probability of a given number of events occurring in a fixed time interval if they are independent and have constant mean. e.g., neutrons at detector, individuals contacting disease.

$X \sim \text{Poisson}(\lambda)$ State Space $E = \{0, 1, 2, \ldots\}$

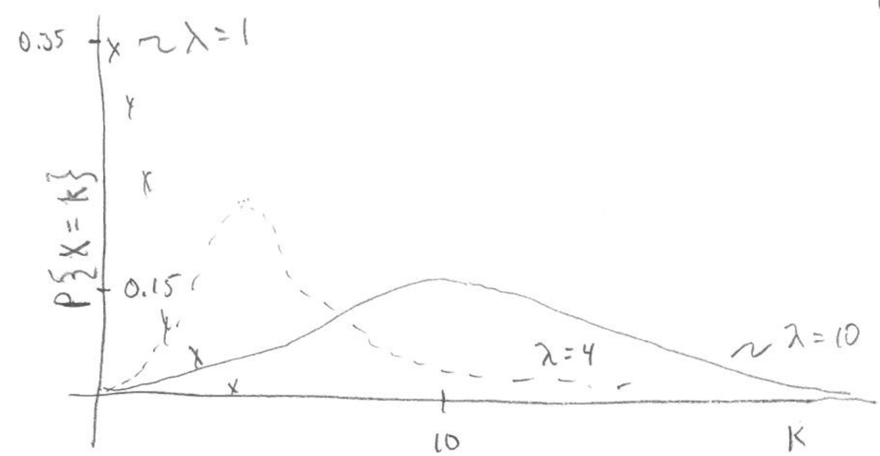
PMF: $P\{X = k \mid \lambda\} = \frac{\lambda^k}{k!} e^{-\lambda}$, $k = 0, 1, \ldots$

Note: $\mathbb{E}[X] = \lambda$, $\text{var}(X) = \lambda$

- For $\lambda \sim 15, 20$, $X \sim N(\lambda, \lambda)$

Note: k: Number of occurrences

λ: Expected rate of occurrences



Property: $X \sim \text{Poisson}(\lambda_x)$

$Y \sim \text{Poisson}(\lambda_Y)$

Then $X + Y \sim \text{Poisson}(\lambda_X + \lambda_Y)$

e.g., Floods occur every 10 years on average. Thus $\lambda = 1$. Then

$P\{k=0\} = \frac{1}{0!} e^{-1} \simeq 0.368$

$P\{k=1\} = \frac{e^{-1}}{1} = 0.368$

$P\{k=2\} = \frac{e^{-1}}{2} \simeq 0.184$

SIR Model: Sum 4 population processes

$$T = \mu N + \mu S + \frac{\beta S I}{N} + (\mu + \gamma) I$$

Interpolation: Rate at which an event, of any type, occurs.

Time between events: $\frac{1}{T}$ so $\underbrace{\lambda = \frac{1}{T}}_{\text{Mean}}$, $P\{X = 1 \mid \lambda\} = \frac{1}{T} e^{-\lambda}$

Strategy: Consider an event to have happened with following probabilities:

$P\{\text{birth}\} = \frac{\mu N}{T}$ $\quad S \to S+1$

$P\{\text{S. Death}\} = \frac{\mu S}{T}$ $\quad S \to S-1$

$P\{\text{Infection}\} = \frac{\beta S I}{N T}$ $\quad S \to S-1,\ I \to I+1$

$P\{\text{Recovery}\} = \frac{\gamma I}{T}$ $\quad I \to I-1$

$P\{\text{Infectious Death}\} = \frac{\mu I}{T}$ $\quad I \to I-1$

1. Sample $X \sim \mathcal{U}(0, T)$: i) Birth if $x < \mu N$
   ii) Susceptible Death if $\mu N < x < \mu N + \mu S$

$\mu N \quad \mu S \quad \frac{\beta S I}{N} \quad (\mu + \gamma) I$

$0 \quad \mu N \quad \mu N + \mu S \quad \mu N + \mu S + \frac{\beta S I}{N} \quad T = \mu N + \mu S + \frac{\beta S I}{N} + (\mu + \gamma) I$

2. Update $S + I$

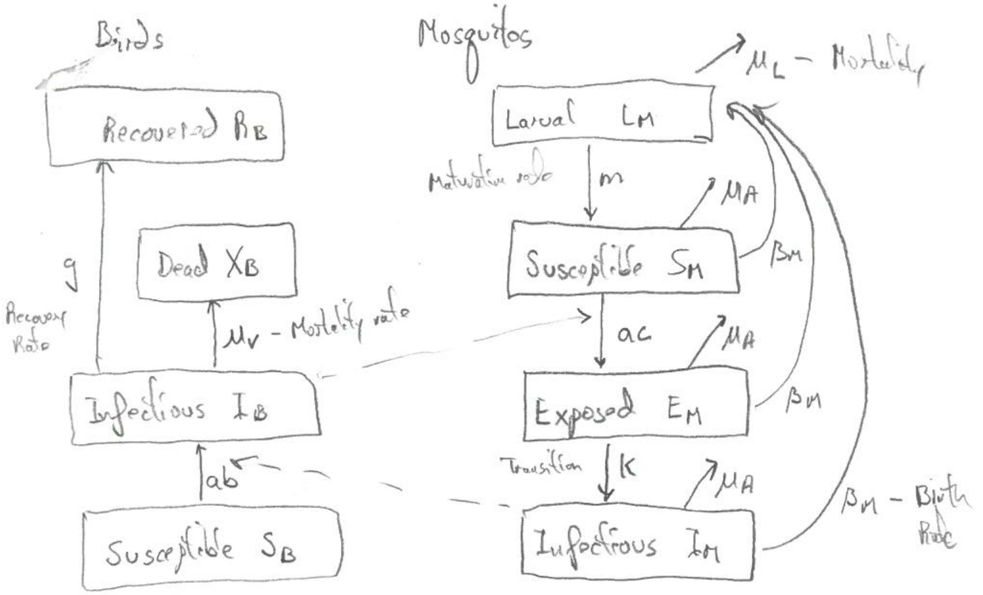
3. Repeat

Note: Attributed to Gillespie but older

$N = 1000$

$\beta = 200$

$\gamma = 100$



$$\frac{dS_B}{dt} = -ab\, I_M \frac{S_B}{N_B}$$

Note: $N_B = S_B + I_B + R_B$

$$\frac{dI_B}{dt} = ab\, I_M \frac{S_B}{N_B} - \mu_V I_B - g I_B$$

$$\frac{dR_B}{dt} = g I_B$$

$$\frac{dX_B}{dt} = \mu_V I_B$$

$$\frac{dL_M}{dt} = \beta_M (S_M + E_M + I_M) - m L_M - \mu_L L_M$$

$$\frac{dS_M}{dt} = -ac\, S_M \frac{I_B}{N_B} + m L_M - \mu_A S_M$$

$$\frac{dE_M}{dt} = ac\, S_M \frac{I_B}{N_B} - k E_M - \mu_A E_M$$

$$\frac{dI_M}{dt} = k E_M - \mu_A I_M$$

Give Table of Values:

Discuss physical versus non-dimensional

e.g., a per capita biting rate on crows

b WN transmission probability, M to B

c WN transmission probability, B to M

Note: ∘ ab, ac appear together

∘ Discuss techniques to isolate parameters

e.g., $y(t) = I_B(t)$

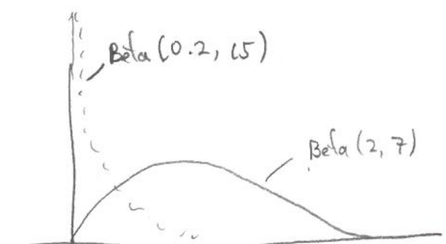
. Consider $\frac{\partial y}{\partial ab}$, $\frac{\partial y}{\partial \mu_V}$, ...

$$\approx \frac{y(g + h_K) - y(g)}{h_K}$$

Parameter Distributions:

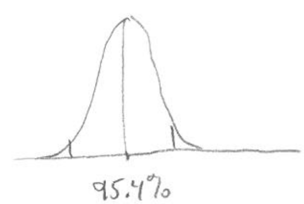
$b \sim U(0.80 - 1)$

$a \sim \mathrm{Beta}(\alpha, \beta)$



$$f_X(x; \alpha, \beta) = \frac{\Gamma(\alpha+\beta)}{\Gamma(\alpha)\Gamma(\beta)} x^{\alpha-1}(1-x)^{\beta-1}$$

$$f_X(x) = \frac{1}{\sigma\sqrt{2\pi}} e^{-(x-\mu)^2/2\sigma^2}, \quad -\infty < x < \infty$$



...dictions: Using mean parameter values, compute

$$S_{m0} = \frac{\mu_a(1+\mu_a)(\gamma+\mu_v)}{\alpha_b \alpha_m} = S_m^* \qquad (*)$$

critical equilibrium mosquito level

Note: $\alpha_b = \frac{ab}{R}$

$\alpha_m = \frac{ac}{R}$

$\mu_a = \frac{\mu_A}{R}$

$\gamma = \frac{g}{R}$

Note: $$R_0 = \sqrt{\frac{ab}{\mu_A} \frac{ac \frac{S_{M0}}{N_{B0}} \left(\frac{R}{R+\mu_A}\right)}{\mu_v + g}} = 1 \qquad (+)$$

Solve for $\frac{S_{M0}}{N_{B0}}$

Note: They report $S_m^* = 4.6$ adult female mosquitoes / initial bird

$R = 0.106$ and min - max values yields $0.37 - 1614$.

Note: Compare $(*)$ and $(+)$