

Population Models

“I keep telling people I don't see explosive population growth. I do see a heck of a lot of building.” Hart Hodges

Motivating Example

Motivation: Use of the mosquitofish *Gambusia Affinis* is a popular means of controlling mosquitos. This methods of control is a multi-million dollar industry world wide. Modeling is required to answer the following questions:

- How many fish should be stocked in each paddy?
- How should the fish be initially stocked? All at once or periodically?
- How should they be stocked to augment the control already provided by endemic fish without highly damaging the local fish populations?



Single Species Model

Notation:

$N(t)$: Number of individuals at time t

β : Constant of proportionality for birth rate

μ : Constant of proportionality for death rate

Model 1:

$$\begin{aligned}\frac{dN}{dt} &= \text{births} - \text{deaths} \\ &= \beta N(t) - \mu N(t) \\ &= \alpha N(t) \\ \Rightarrow N(t) &= N_0 e^{\alpha t}\end{aligned}$$

(i) $\alpha > 0$ Exponential growth

(ii) $\alpha < 0$ Exponential decay

Limitations: Does not incorporate overcrowding, depletion of resources, predators, size effects, etc.

Single Species Model

Model 2: (Logistic model) Death rate is population dependent

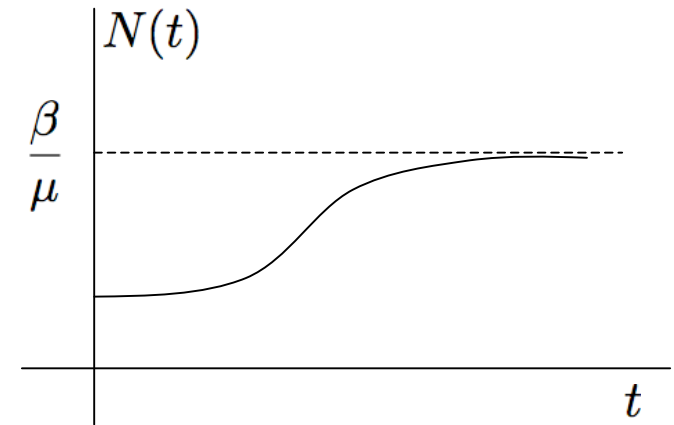
$$\frac{dN}{dt} = \beta N(t) - [\mu N(t)]N(t)$$

Behavior: Let $\frac{dN}{dt} = 0$ so that

$$N(t) = 0 \text{ or } N(t) = \frac{\beta}{\mu}$$

Solution:
$$N(t) = \frac{\beta}{\mu + (\beta - \mu N_0) \frac{1}{N_0} e^{-\beta t}}$$

$$\Rightarrow \lim_{t \rightarrow \infty} N(t) = \frac{\beta}{\mu}$$



Limitations: Does not incorporate predators or size effects.

Predator-Prey Model

Notation:

$N(t)$: Number of predators at time t

$E(t)$ Number of prey at time t

Model: Lotka (1925), Volterra (1926)

$$\frac{dE}{dt} = \beta_E E(t) - [\mu_E N(t)] E(t)$$

$$\frac{dP}{dt} = [\beta_N E(t)] N(t) - \mu_N N(t)$$

Equilibria:

$$\frac{dE}{dt} = \frac{dN}{dt} = 0$$

$$\Rightarrow N^e = \frac{\beta_E}{\mu_E}, E^e = \frac{\mu_N}{\beta_N} \text{ or } N^e = E^e = 0$$

Predator-Prey Model

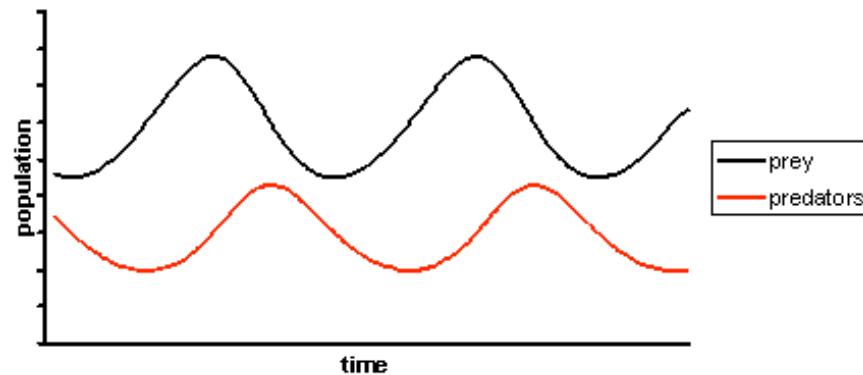
Note:
$$\frac{dN}{dE} = \frac{N(-\mu_N + \beta_N E)}{E(\beta_E - \mu_E N)}$$

$$\Rightarrow \int \frac{\beta_E - \mu_E N}{N} dN = \int \frac{-\mu_N + \beta_N E}{E} dE$$

$$\Rightarrow \beta_E \ln N - \mu_E N = -\mu_N \ln E + \beta_N E + \ln k_1$$

$$\Rightarrow \frac{N^{\beta_E} E^{\mu_N}}{e^{\mu_E N} e^{\beta_N E}} = k_1$$

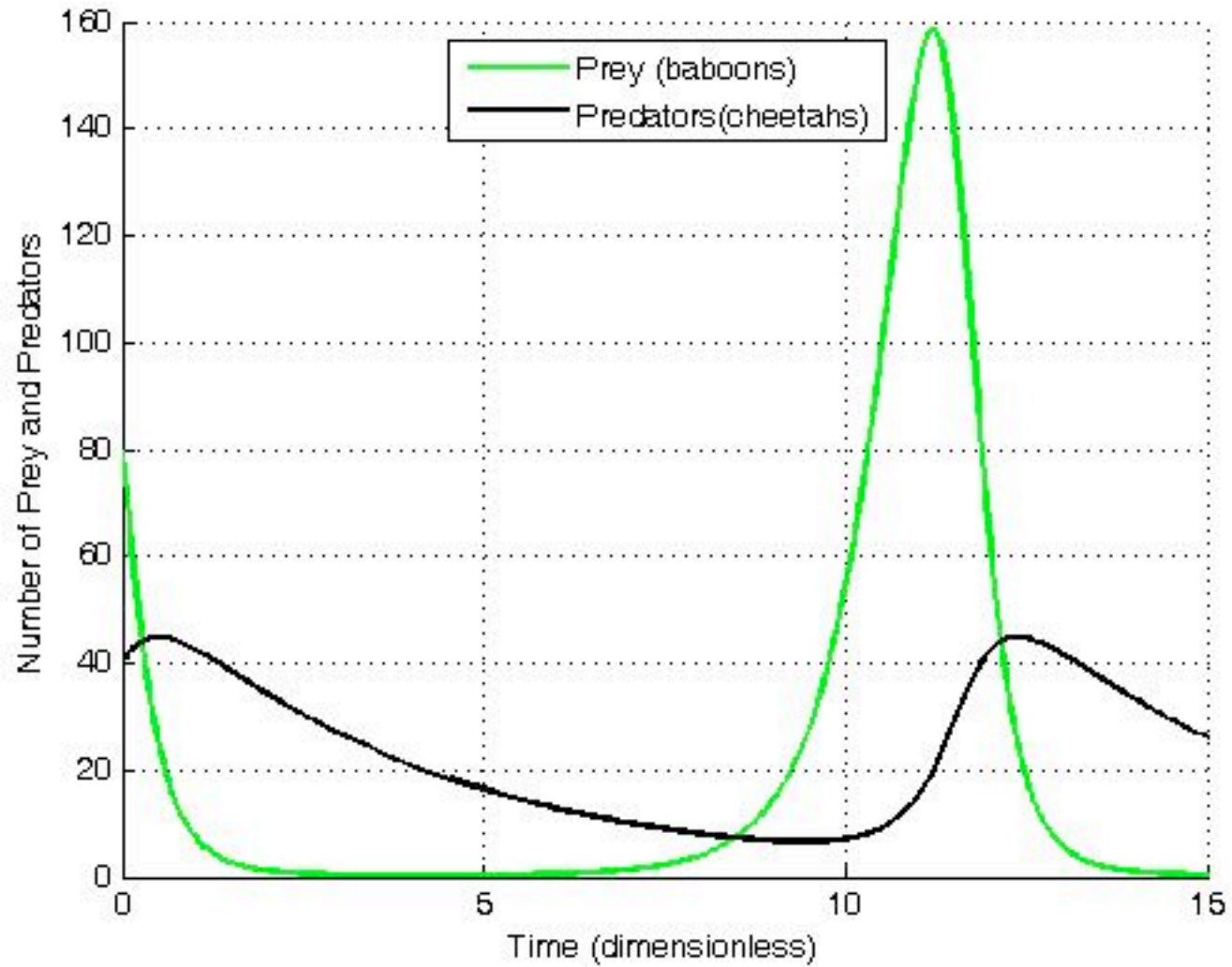
Solution Behavior:



Limitations: Does not incorporate size effects, gender differences, outside influences (e.g., effects of trapping).

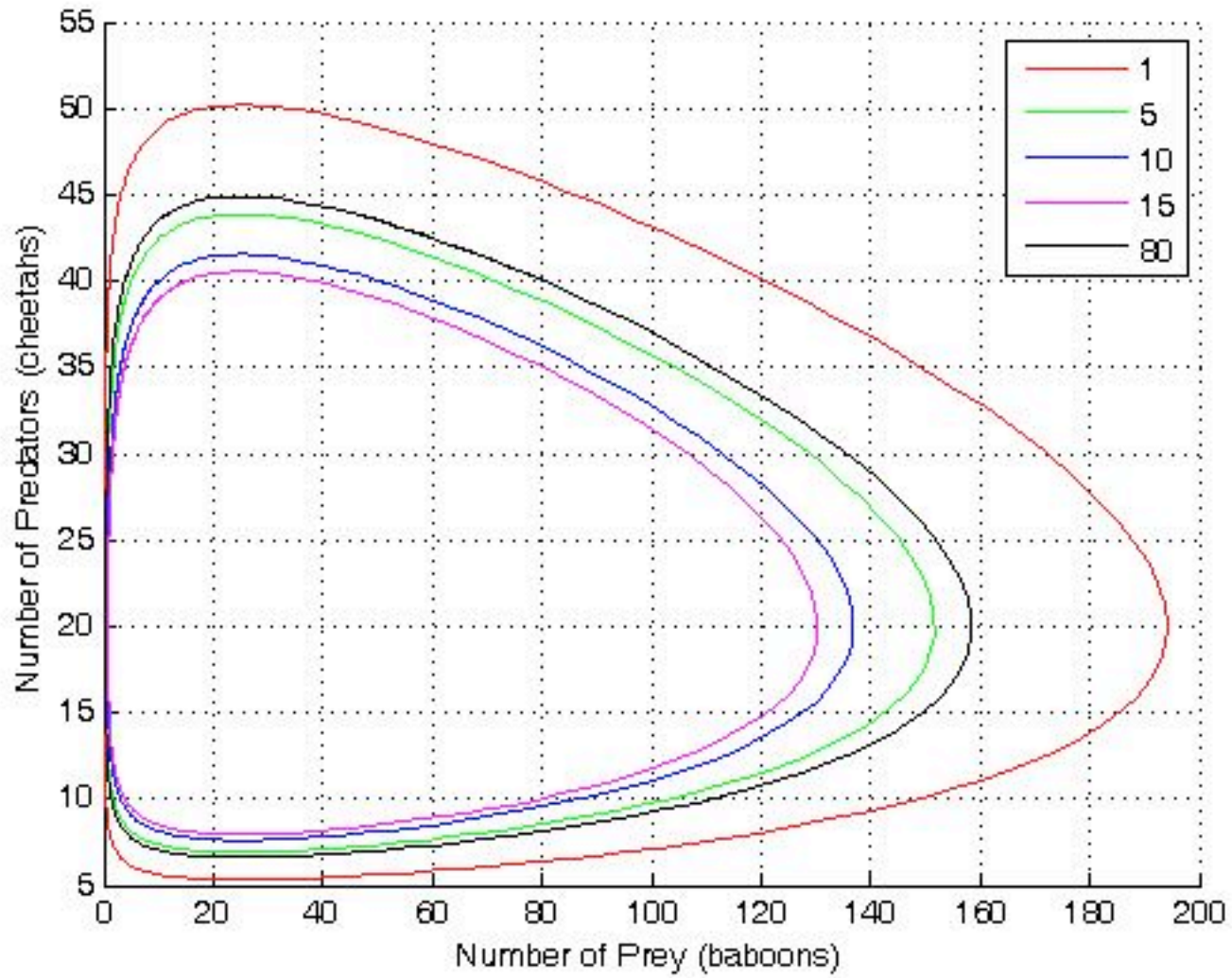
Predator-Prey Model

Example:



Predator-Prey Model

Example:

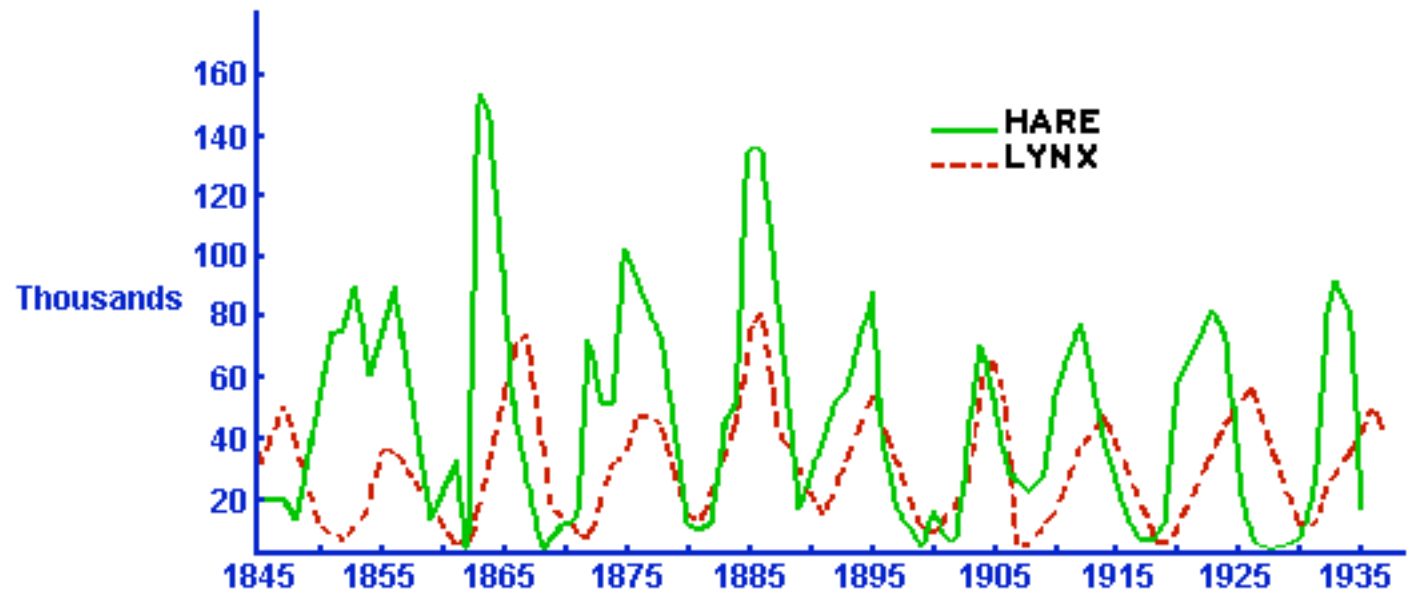


Predator-Prey Model

Example: Lynx-Hare data collected by the Hudson Bay Company



Question: Why does the hare population lag behind the lynx population? See “Do Hares Eat Lynx”, *The American Naturalist*, 107(957), pp. 727-730, 1973.



Size-Structured Models

Note: The logistic model and Lotka-Volterra predator-prey models are aggregate in the sense that they consider total populations in which individuals are assumed to have identical characteristics. Introducing individual characteristics is much more difficult. Sinko and Streifer balanced the two approaches by assuming that individuals share common traits.

Notation: $u(t, x)$ Number of individuals of size x at time t

$N(t)$ Number of total individuals

Note:

$$N(t) = \sum_{i=1}^M u(t, x_i) \quad , \quad M \text{ discrete sizes}$$

$$N_{a,b}(t) = \int_a^b u(t, \xi) d\xi \quad , \quad \text{Number of individuals between sizes } a \text{ and } b \text{ at time } t$$

Sinko-Streifer Theory

Assumptions:

(1) Growth rate of same sized individuals is the same and is denoted by g

$$\frac{dx}{dt} = g(t, x)$$

(2) Same sized individuals have same likelihood of death

$$\frac{dN}{dt} = \mu N(t)$$

(3) There is a smallest size x_0 and largest size x_1

(4) The birth rate is proportional to the population size

$$R(t) = \int_{x_0}^{x_1} k(t, \xi) u(t, \xi) d\xi$$

(5) Population is sufficiently large to permit continuum model

Sinko-Streifer Model

Case 1: No deaths

Flux Balance: Note that $u(t, x)$ is a “density” and the rate is $q(t, x) = g(t, x)u(t, x)$

$$\begin{aligned}\frac{\partial u}{\partial t} + \frac{\partial q}{\partial x} &= 0 \\ \Rightarrow \frac{\partial u}{\partial t} + \frac{\partial(gu)}{\partial x} &= 0\end{aligned}$$

First Principles Derivation: Let $x(t; t_0, \eta)$ denote the solution to

$$\begin{aligned}\dot{x}(t) &= g(t, x) \\ x(t_0) &= \eta\end{aligned}$$

Note: Individuals cannot jump between sizes

$$\Rightarrow N_{a,b}(t_0) = N_{x(t;t_0,a),x(t;t_0,b)}(t)$$

Sinko-Streifer Model

Thus

$$\begin{aligned}\int_a^b u(t_0, \xi) d\xi &= \int_{x(t;t_0,a)}^{x(t;t_0,b)} u(t, \xi) d\xi \\ \Rightarrow 0 &= \frac{d}{dt} \int_{x(t;t_0,a)}^{x(t;t_0,b)} u(t, \xi) d\xi \\ &= \int_{x(t;t_0,a)}^{x(t;t_0,b)} \frac{\partial u}{\partial t} d\xi + u(t, x(t; t_0, b)) \dot{x}(t; t_0, b) - u(t, x(t; t_0, a)) \dot{x}(t; t_0, a) \\ &= \int_{x(t;t_0,a)}^{x(t;t_0,b)} \frac{\partial u}{\partial t} d\xi + u(t, x(t; t_0, b)) g(t, x(t; t_0, b)) - u(t, x(t; t_0, a)) g(t, x(t; t_0, a)) \\ &= \int_{x(t;t_0,a)}^{x(t;t_0,b)} \frac{\partial u}{\partial t} d\xi + \int_{x(t;t_0,a)}^{x(t;t_0,b)} \frac{\partial}{\partial \xi} [u(t, \xi) g(t, \xi)] d\xi \\ \Rightarrow \frac{\partial u}{\partial t} + \frac{\partial(gu)}{\partial \xi} &= 0\end{aligned}$$

Sinko-Streifer Model

Case 2: Include deaths using flux balance

$$\frac{\partial u}{\partial t} + \frac{\partial(gu)}{\partial x} = -\mu u$$

Births: Assume they act as flux in (Boundary Condition) at size x_0 . Recall that the rate is $q = gu$ so that

$$R(t) = g(t, x)u(t, x)|_{x=x_0} = \int_{x_0}^{x_1} k(t, \xi)u(t, \xi)d\xi$$

Initial Condition: $u(0, x) = \Phi(x)$

Model:

$$\frac{\partial u}{\partial t} + \frac{\partial(gu)}{\partial x} = -\mu u$$

$$g(t, x)u(t, x)|_{x=x_0} = \int_{x_0}^{x_1} k(t, \xi)u(t, \xi)d\xi$$

$$u(0, x) = \Phi(x)$$

Sinko-Streifer Model

Solution: Determined by characteristics

Simplified Example: Take $g(t, x) = a$ and $\mu(t, x) = 0$

$$\frac{\partial u}{\partial t} + a \frac{\partial u}{\partial x} = 0$$

$$u(0, x) = \Phi(x)$$

Solution: Recall from the notes on analytic solutions for the wave equation

$$u(t, x) = \Phi(x - at)$$

Total Derivative:

$$\frac{du}{dt} = u_t + u_x \frac{dx}{dt}$$

Characteristic Equation:

$$\begin{aligned} \frac{dx}{dt} &= a & \Rightarrow x(t) &= at + x_0 \\ x(0) &= x_0 \end{aligned}$$

Note: $\frac{du}{dt} = 0$

$\Rightarrow u(t, x)$ is constant
along characteristics

Sinko-Streifer Model

Model:

$$\frac{\partial u}{\partial t} + \frac{\partial(gu)}{\partial x} = -\mu u$$

$$g(t, x_0)u(t, x_0) = R(t)$$

$$u(0, x) = \Phi(x)$$

Characteristic Curve: $\frac{dx}{dt} = g$. Thus

$$\begin{aligned}\frac{du}{dt} &= \frac{\partial u}{\partial x} \frac{dx}{dt} + \frac{\partial u}{\partial t} \\ &= \frac{\partial u}{\partial x} g(t, x) + \frac{\partial u}{\partial t} \\ &= -u(t, x) \frac{\partial}{\partial x} g(t, x) - \mu(t, x)u(t, x)\end{aligned}$$

Along characteristic curve, Sinko-Streifer solution satisfies

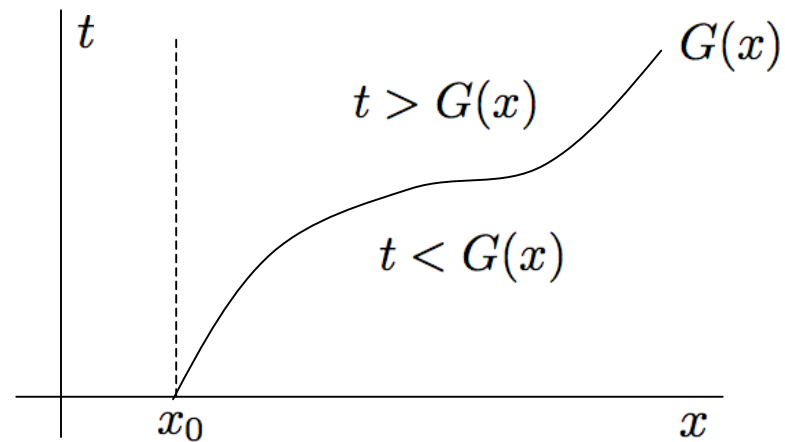
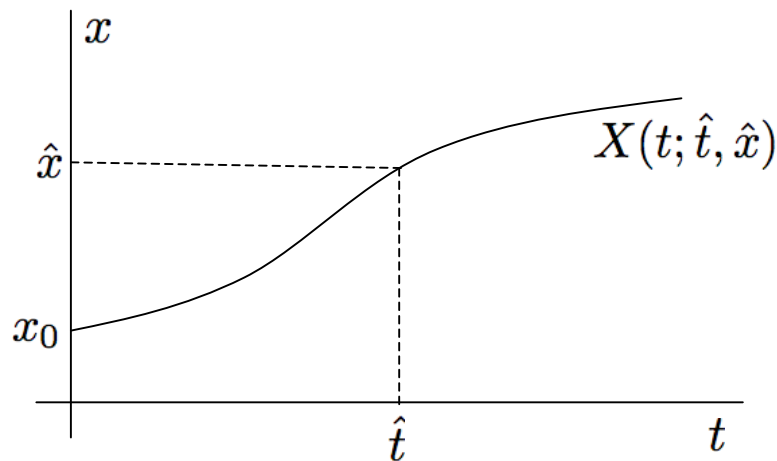
$$\begin{aligned}\frac{du}{dt} &= -(g_x + \mu)u \\ \Rightarrow u &= v_0 e^{-\int (g_x + \mu) dt}\end{aligned}$$

Sinko-Streifer Model

Notation: Let $(t, X(t; \hat{t}, \hat{x}))$ denote characteristic curve through (\hat{t}, \hat{x}) where

$$\frac{d}{dt} X(t; \hat{t}, \hat{x}) = g(t, X(t; \hat{t}, \hat{x}))$$

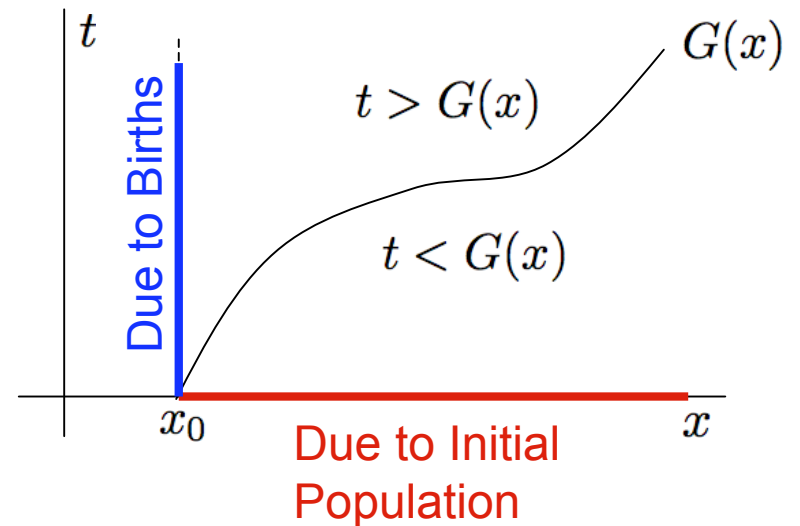
$$X(\hat{t}; \hat{t}, \hat{x}) = \hat{x}$$



Note: $g > 0 \Rightarrow \frac{dX}{dt} > 0$ so X has an inverse $T(x; \hat{t}, \hat{x}) : x \rightarrow t$. Let

$G(x) = T(x; 0, x_0)$ denote curve through $(0, x_0)$.

Sinko-Streifer Model



Case 1: $t \leq G(x)$

$$u(t, x) = v_0 e^{-\int_0^t [g_x(\tau, x) + \mu(\tau, x)] d\tau}$$

$$\Rightarrow u(0, x) = v_0 = \Phi(x)$$

$$\Rightarrow u(t, x) = \Phi(X(0; t, x)) e^{-\int_0^t [g(\tau, X(\tau; t, x)) + \mu(\tau, X(\tau; t, x))] d\tau}$$

Case 2: $t > G(x)$

$$u(t, x) = v_0 e^{-\int_{T(x_0; t, x)}^t [g_x(\tau, x) + \mu(\tau, x)] d\tau}$$

$$\Rightarrow u(t_0, x_0) = v_0 e^{-\int_{T_0}^{t_0} [g_x(\tau, x_0) + \mu(\tau, x_0)] d\tau} = \frac{R(t_0)}{g(t_0, x_0)}$$

$$\Rightarrow u(t, x) = \frac{R(T(x_0; t, x))}{g(T(x_0; t, x))} e^{-\int_{T(x_0; t, x)}^t [g(\tau, X(\tau; t, x)) + \mu(\tau, X(\tau; t, x))] d\tau}$$

Sinko-Streifer Model

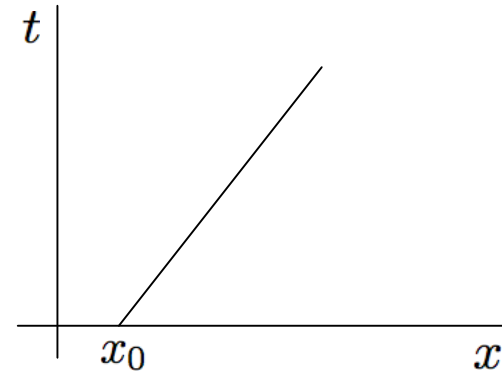
Example: Constant growth and mortality: $g = g_0, \mu = \mu_0$

Model:

$$\frac{\partial u}{\partial t} + g_0 \frac{\partial u}{\partial x} = -\mu_0 u$$

$$g_0 u(t, x_0) = R(t)$$

$$u(0, x) = \Phi(x)$$



Characteristic Curve:

$$\frac{dx}{dt} = g_0$$

$$x(\hat{t}) = \hat{x}$$

$$\Rightarrow x - \hat{x} = g_0(t - \hat{t})$$

$$\Rightarrow X(t; \hat{t}, \hat{x}) = \hat{x} + g_0(t - \hat{t})$$

$$\Rightarrow T(x; \hat{t}, \hat{x}) = \hat{t} + \frac{1}{g_0}(x - \hat{x})$$

Curve through (x, x_0) :

$$G(x) = T(x; 0, x_0) = \frac{1}{g_0}(x - x_0)$$

Initial Condition Driven Solution:

$$\begin{aligned} u(t, x) &= \Phi(x - g_0 t) e^{-\int_0^t \mu_0 d\tau} \\ &= \Phi(x - g_0 t) e^{-\mu_0 t} \end{aligned}$$

Sinko-Streifer Model

Birth Driven Solution:

$$\begin{aligned} u(t, x) &= \frac{R\left(t + \frac{1}{g_0}(x_0 - x)\right)}{g_0} e^{-\int_{t + \frac{1}{g_0}(x_0 - x)}^t \mu_0 d\tau} \\ &= \frac{R\left(t + \frac{1}{g_0}(x_0 - x)\right)}{g_0} e^{\frac{\mu_0}{g_0}(x_0 - x)} \end{aligned}$$

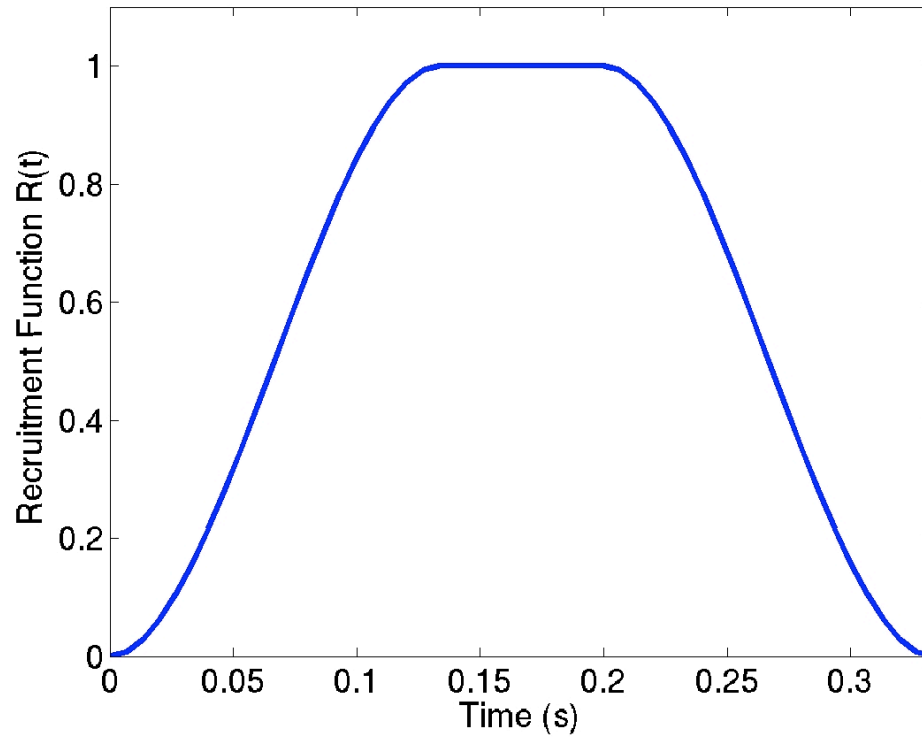
Example: Consider

$$R(t) = \begin{cases} 3/4[(\alpha t - 1) - (\alpha t - 1)^3/3 + 2/3], & t \in [0, 2/\alpha], \\ 1, & t \in (2/\alpha, 2/\alpha + \beta), \\ -3/4[(\alpha s - 1) - (\alpha s - 1)^3/3 - 2/3], & s = t - 2/\alpha - \beta, \\ & t \in [2/\alpha + \beta, 4/\alpha + \beta], \\ 0, & \text{otherwise} \end{cases}$$

with $\alpha = 15$ and $\beta = 1/\alpha$

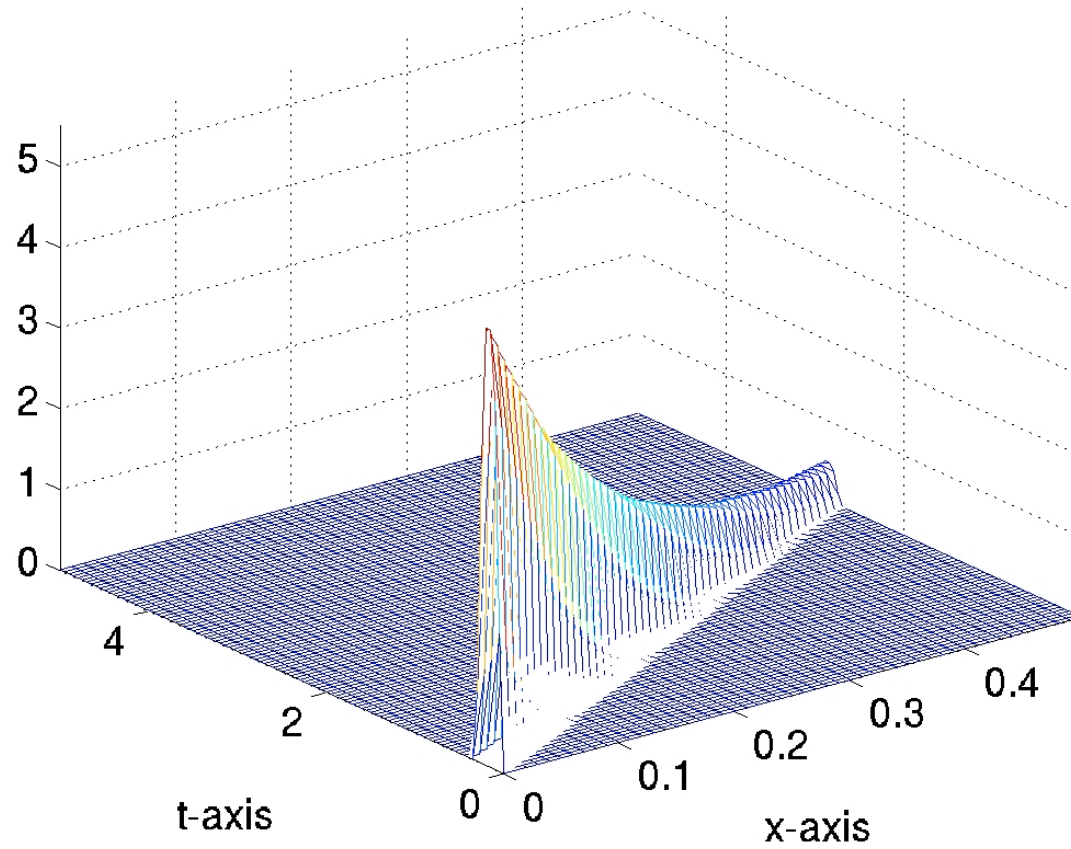
Sinko-Streifer Model

Example: Recruitment function



Sinko-Streifer Model

Example: Consider $\Phi(x) = 0, g_0 = 0.185, \mu_0 = 0.9$



Sinko-Streifer Model

Example: Consider $\Phi(x) = 0, g_0 = 0.185, \mu_0 = 0.9$

