Heat Conduction and the Heat Equation

"If you can't take the heat, don't tickle the dragon."

Heat Transfer

Note: Energy is the conserved quantity

Conduction:

- Heat transfer due to molecular activity. Energy is transferred from more energetic to less energetic particles due to energy gradient
- Occurs in solids, fluids and gases
- Empirical relation: Fourier's law

Convection:

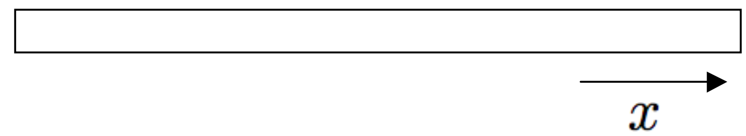
- Energy transfer in fluid or gas due to bulk or macroscopic motion (advection)
- Convection: Advection + conduction
- Empirical relation: Newton's law of cooling

Thermal Radiation:

- Energy emitted by matter due to changes in electron configurations that results in changes in energy via EM waves or photons
- Empirical relation: Stefan-Boltzmann law

Heat Conduction

1-D Assumptions:



- Temperature uniform over cross-sections
- Heat transfer is by conduction
- Heat transfer only along x-axis
- No heat escapes from sides (perfect insulation)

Relevant Quantities:

- $u(t, x)$: Temperature (^{o}C or K)
- $H(t, x)$: Amount of Heat (energy) – Units: Calories or Joules

  Note: 1 calorie = heat required to raise 1 g water 1 ^{o}C

  1 J = 0.23885 cal

  1 cal = 4.19 J

Note: $H = c_p m u$

- c_p: Specific heat – Units: $\frac{\mathrm{J}}{\mathrm{kg} \cdot \mathrm{K}}$, e.g., Aluminum versus iron
- m: Mass , e.g., Consider 1 g Fe at 100 ^{o}C in 10 g water versus

  10 g Fe at 100 ^{o}C in 10 g water

Heat Conduction

Thermal Energy Density: Units: J/m^3 or cal/m^3

$$\rho_{th}(t,x) = H(t,x)/m^3 = c_p \rho(t,x) u(t,x)$$

Rate of Heat Transfer:

$q(t,x)$: Units: Watts (power) where 1 W = 1 J/s

Flux:

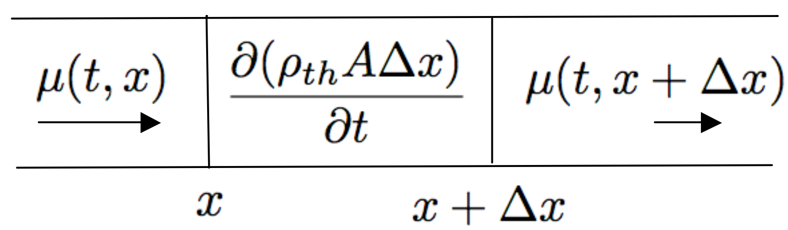
$$\mu(t,x) = q(t,x)/A$$

Conservation of Energy: See mass conservation with constant A

$$\frac{\partial \rho_{th}}{\partial t} + \frac{\partial \mu}{\partial x} = 0$$

$$\boxed{\Rightarrow c_p \rho \frac{\partial u}{\partial t} = -\frac{\partial \mu}{\partial x}}$$

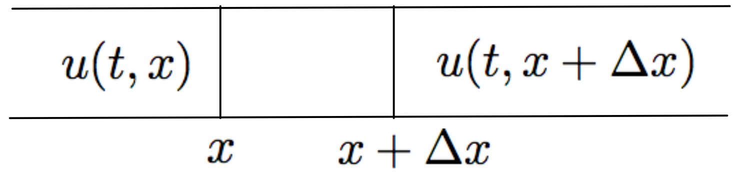
if c_p and ρ are constant



Note: $c_p \rho$: Volumetric heat capacity
(ability of material to store heat)

1-D Heat Equation

Constitutive Relation:



$$q = kA\frac{u(t,x) - u(t,x+\Delta x)}{\Delta x}$$

$$\boxed{\Rightarrow q = -kA\frac{\partial u}{\partial x}}$$

Fourier's law of heat conduction

or

$$\boxed{\mu = -k\frac{\partial u}{\partial x}}$$

Note: k provides measure of material's ability to conduct heat

1-D Unforced Heat Equation:

$$\boxed{\frac{\partial u}{\partial t} = \alpha\frac{\partial^2 u}{\partial x^2}}$$

Note: Thermal diffusivity $\alpha = \dfrac{k}{\rho c_p}$ has units $\frac{m^2}{s}$

Boundary and Initial Conditions

Initial Condition:

$$u(0,x) = \psi(x)\ ,\ 0 < x < L$$

Dirichlet Boundary Condition: Specify temperature

$$u(t,0) = u_1(t)\ ,\ u(t,L) = u_2(t)$$

Neumann Boundary Condition: Specify $q(t,x)$ or $\mu(t,x)$ at boundaries

e.g., Perfectly insulated at $x = 0$ implies $k\dfrac{\partial u}{\partial x}(t,0) = 0$

Robin Boundary Condition:

e.g., $k\dfrac{\partial u}{\partial x}(t,L) + hu(t,L) = g(t)$

Boundary Conditions

Robin Boundary Condition: Motivation

Newton's Law of Cooling:

$$\boxed{q = hA(u_s - u_f)}$$

u_s – Temperature of solid

u_f – Temperature of fluid or convective medium

h – Convective heat transfer coefficient

Mechanism	$h\ (W/m^2K)$
Still air	2.8-23
Moving air	11.3-55
Moving water	280-17,000
Condensing steam	5,700-28,000

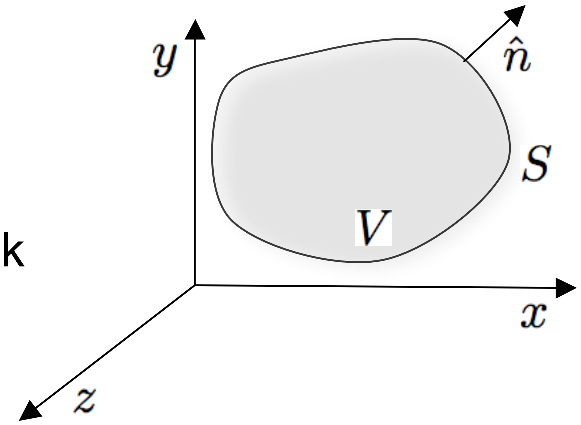
Note:

$$-\mu(t,L) = h[u_f - u(t,L)]$$

$$\Rightarrow k\frac{\partial u}{\partial x}(t,L) = h[u_f - u(t,L)]$$

$$\Rightarrow k\frac{\partial u}{\partial x}(t,L) + hu(t,L) = hu_f$$

3-D Heat Equation



Fourier's Law:

$$\mu = -k\nabla u \cdot \hat{n}$$

Conservation of Energy: Let f denote heat source or sink

$$\frac{d}{dt}\int_V \rho_{th}\, dV = -\int_S \mu\, dS + \int_V f\, dV$$

$$\Rightarrow \int_V c_p \rho \frac{\partial u}{\partial t} = \int_S k\nabla u \cdot \hat{n}\, dS + \int_V f\, dV$$

$$\Rightarrow \int_V \left(c_p \rho \frac{\partial u}{\partial t} - \nabla \cdot (k\nabla u) - f \right) dV = 0$$

$$\boxed{\Rightarrow c_p \rho \frac{\partial u}{\partial t} = k\nabla^2 u + f}$$ if k is constant

Note: $\nabla^2 u = \Delta u = \frac{\partial^2 u}{\partial x^2} + \frac{\partial^2 u}{\partial y^2} + \frac{\partial^2 u}{\partial z^2}$ in cartesian coordinates

Thermal-Based Damage Detection in Porous Materials

Current Research: H.T. Banks, Amanda Criner (NCSU), William Winfree (NASA LaRC)

Detail: CRSC Technical Report CRSC-TR08-11

Goal: Use active thermography to detect subsurface anomalies in porous materials; e.g., for aeronautic and aerospace structures

Homogenized Model:

$$\frac{\partial u}{\partial t} - k\Delta u = 0 \text{ in } \ \Omega$$

$$\left.\frac{\partial u}{\partial \eta}\right|_{\cup_i \partial\Omega_i} = \left.\frac{\partial u}{\partial \eta}\right|_{\cup_{i=1}^{3}\omega_i} = u|_{\omega_5} = 0$$

$$\left.\frac{\partial u}{\partial \eta}\right|_{\omega_4} = S_0 \chi_{[t_0,t_s]}(t)$$

$$u(0,x) = u_0$$