Analytic Techniques for Advection-Diffusion Equations

"Furious activity is no substitute for understanding," H.H. Williams

Properties of Fourier Series

Fourier Series: Consider the representation

$$f(x) = \frac{a_0}{2} + \sum_{n=1}^{\infty} \left[a_n \cos\left(\frac{n\pi x}{L}\right) + b_n \sin\left(\frac{n\pi x}{L}\right) \right]$$

for a $2L$ periodic, continuous function $f(x)$. If the series converges uniformly, it is the Fourier series for $f(x)$.

Note:

$$\int_{-L}^{L} f(x)dx = a_0 L$$

$$\int_{-L}^{L} \sin\left(\frac{m\pi x}{L}\right) \sin\left(\frac{n\pi x}{L}\right) dx = \int_{-L}^{L} \cos\left(\frac{m\pi x}{L}\right) \cos\left(\frac{n\pi x}{L}\right) dx = 0 \quad \text{for } m \neq n$$

$$\int_{-L}^{L} \sin\left(\frac{m\pi x}{L}\right) \cos\left(\frac{n\pi x}{L}\right) dx = 0 \quad \text{for all } m, n$$

$$\int_{-L}^{L} \sin^2\left(\frac{m\pi x}{L}\right) dx = \cos^2\left(\frac{n\pi x}{L}\right) dx = L \quad \text{for } n \geq 1$$

Properties of Fourier Series

Fourier Coefficients:

$$a_n = \frac{1}{L}\int_{-L}^{L} f(x)\cos\left(\frac{n\pi x}{L}\right)dx \quad \text{for } n = 0, 1, 2, \ldots$$

$$b_n = \frac{1}{L}\int_{-L}^{L} f(x)\sin\left(\frac{n\pi x}{L}\right)dx \quad \text{for } n = 1, 2, \ldots$$

Theorem: Suppose that f and f' are piecewise continuous on the interval $-L \leq x \leq L$. Further, suppose that f is defined outside the interval $-L \leq x \leq L$ so that it is periodic with period $2L$. Then f has a Fourier series

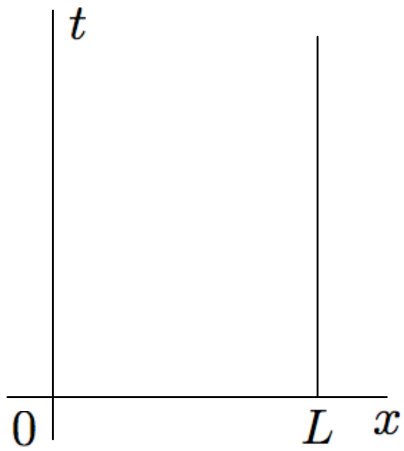
$$f(x) = \frac{a_0}{2} + \sum_{n=1}^{\infty}\left[a_n \cos\left(\frac{n\pi x}{L}\right) + b_n \sin\left(\frac{n\pi x}{L}\right)\right]$$

with the above coefficients. The Fourier series converges to $f(x)$ at all points where f is continuous, and to $[f(x+) + f(x-)]/2$ at all points where f is discontinuous (see Boyce and DiPrima).

Analytic Solution Techniques: Diffusion Equation

Recall: General advection-diffusion equation

$$\frac{\partial \rho}{\partial t} + \frac{\partial(\rho \bar{u})}{\partial x} = \frac{\partial}{\partial x}\left(D\frac{\partial \rho}{\partial x}\right) + b - d$$

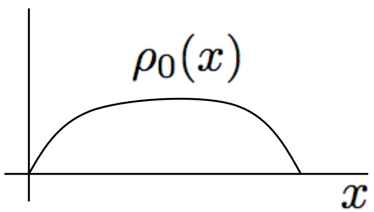


Diffusion Equation: Consider $\bar{u} = b = d = 0$

$$\frac{\partial \rho}{\partial t} = \frac{\partial}{\partial x}\left(D\frac{\partial \rho}{\partial x}\right)$$

$$\rho(t,0) = \rho_L(t)\ ,\ \rho(t,L) = \rho_R(t)$$

$$\rho(0,x) = \rho_0(x)$$



Simplifying Assumption: D constant, $\rho_L(t) = \rho_R(t) = 0$

Analytic Solution Techniques: Diffusion Equation

Separation of Variables: Consider solutions of the form

$$\rho(t,x) = X(x)T(t)$$

$$\Rightarrow X(x)\dot{T}(t) = DX''(x)T(t)$$

$$\Rightarrow \frac{X''(x)}{X(x)} = \frac{\dot{T}(t)}{DT(t)} = c$$

Hence

$$X''(x) - cX(x) = 0 \qquad \dot{T}(t) = cDT(t)$$

$$X(0) = X(L) = 0 \qquad \text{and} \qquad \Rightarrow T(t) = \alpha e^{cDt}$$

Note:

$$\int_0^L \left[XX'' - cX^2\right] dx = -\int_0^L \left[-(X')^2 - cX^2\right] dx = 0$$

If $c \geq 0$, this implies that $X(x) = k = 0$. Thus $c < 0$ so take $c = -\lambda^2, \lambda > 0$.

Analytic Solution Techniques: Diffusion Equation

Boundary Value Problem:

$$X''(x) + \lambda^2 X(x) = 0$$

$$X(0) = X(L) = 0$$

Solution: $X(x) = A\cos(\lambda x) + B\sin(\lambda x)$

$$X(0) = 0 \Rightarrow A = 0$$

$$X(L) = 0 \Rightarrow \lambda L = n\pi$$

Thus

$$X_n(x) = B_n \sin(\lambda_n x) \quad , \ \lambda_n = \frac{n\pi}{L} \ , B_n \neq 0$$

PDE Solution:

$$\rho(t,x) = \sum_{n=1}^{\infty} \alpha_n e^{-D\lambda_n^2 t} \sin(\lambda_n x)$$

Analytic Solution Techniques: Diffusion Equation

Initial Condition:

$$\rho(0,x) = \sum_{n=1}^{\infty} \alpha_n \sin(\lambda_n x) = \rho_0(x)$$

$$\Rightarrow \alpha_n = \frac{2}{L}\int_0^L \rho_0(x)\sin(\lambda_n x)dx$$

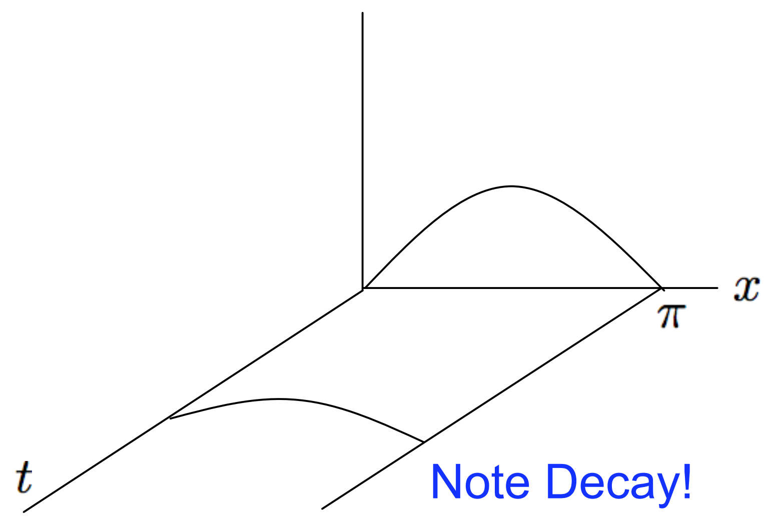
Example: $\rho_0(x) = \sin(\pi x/L)$

$$\alpha_n = \frac{2}{L}\int_0^L \sin\left(\frac{\pi x}{L}\right)\sin\left(\frac{n\pi x}{L}\right)dx$$

$$= \begin{cases} \frac{2}{L}\cdot\frac{L}{2} & , \quad n=1 \\ 0 & , \quad n \neq 1 \end{cases}$$

Solution

$$\rho(t,x) = e^{-D(\pi/L)^2 t}\sin(\pi x/L)$$



Analytic Solution Techniques: Diffusion Equation

Forced Response:

$$\rho_t = D\rho_{xx} + f(t,x)$$

$$\rho(t,0) = \rho(t,L) = 0$$

$$\rho(0,x) = \rho_0(x)$$

Consider solution of the form (variation of parameters)

$$\rho(t,x) = \sum_{n=1}^{\infty} \alpha_n(t) e^{-D\lambda_n^2 t} \sin(\lambda_n x)$$

Expand force:

$$f(t,x) = \sum_{n=1}^{\infty} f_n(t) e^{-D\lambda_n^2 t} \sin(\lambda_n x)$$

where

$$f_n(t) e^{-D\lambda_n^2 t} = \frac{2}{L} \int_0^L f(t,x) \sin(\lambda_n x)$$

Analytic Solution Techniques: Diffusion Equation

Assumption: Piecewise differentiation yields

$$\sum_{n=1}^{\infty}\left[\alpha_n'(t) - D\lambda_n^2\alpha_n(t) + D\lambda_n^2\alpha_n(t) - f_n(t)\right] e^{-D\lambda_n^2 t}\sin(\lambda_n x) = 0$$

$$\Rightarrow \alpha_n'(t) = f_n(t)$$

$$\Rightarrow \alpha_n(t) = \alpha_n + \int_0^t f_n(\tau)d\tau$$

since $\alpha_n(0) = \frac{2}{L}\int_0^L \rho_0(x)\sin(\lambda_n x)dx = \alpha_n$

Formal Solution:

$$\rho(t,x) = \sum_{n=1}^{\infty}\alpha_n e^{-D\lambda_n^2 t}\sin(\lambda_n x) + \sum_{n=1}^{\infty}\left[\int_0^t f_n(\tau)d\tau\right] e^{-D\lambda_n^2 t}\sin(\lambda_n x)$$

Analytic Solution Techniques: Diffusion Equation

Special Case: Periodic source $f(t,x) = f(x)e^{i\omega t}$

Thus

$$f_n(t)e^{-D\lambda_n^2 t} = \frac{2}{L}\int_0^L f(x)e^{i\omega t}\sin(\lambda_n x)dx = \mathcal{F}_n e^{i\omega t}$$

where

$$\mathcal{F}_n = \frac{2}{L}\int_0^L f(x)\sin(\lambda_n x)dx$$

Soo ...

$$f_n(t) = \mathcal{F}_n e^{(D\lambda_n^2 + i\omega)t} \Rightarrow \int_0^t f_n(\tau)d\tau = \frac{\mathcal{F}_n}{D\lambda_n^2 + i\omega}\left[e^{(D\lambda_n^2+i\omega)t} - 1\right]$$

Formal Solution:

$$\rho(t,x) = \sum_{n=1}^{\infty}\alpha_n e^{-D\lambda_n^2 t}\sin(\lambda_n x) - \sum_{n=1}^{\infty}\frac{\mathcal{F}_n}{D\lambda_n^2 + i\omega}e^{-D\lambda_n^2 t}\sin(\lambda_n x)$$

$$+\sum_{n=1}^{\infty}\frac{\mathcal{F}_n}{D\lambda_n^2 + i\omega}e^{i\omega t}\sin(\lambda_n x)$$ ← Steady State Solution

Green's Function

Consider:

$$\rho_t = D\rho_{xx}$$

$$\rho(t,0) = \rho(t,L) = 0$$

$$\rho(0,x) = \rho_0(x)$$

Solution:

$$\rho(t,x) = \sum_{n=1}^{\infty} \alpha_n e^{-D\lambda_n^2 t} \sin(\lambda_n x) \quad , \quad \alpha_n = \frac{2}{L}\int_0^L \rho_0(y) \sin(\lambda_n y dy$$

With uniform convergence, we can write

$$\begin{aligned} \rho(t,x) &= \int_0^L \left[\frac{2}{L} \sum_{n=1}^{\infty} e^{-D\lambda_n^2 t} \sin(\lambda_n x) \sin(\lambda_n y) \right] \rho_0(y) dy \\ &= \int_0^L G(x,y,t) \rho_0(y) dy \end{aligned}$$

Green's Function:

$$G(x,y,t) = \frac{2}{L} \sum_{n=1}^{\infty} e^{-D\lambda_n^2 t} \sin(\lambda_n x) \sin(\lambda_n y)$$

Analogy:

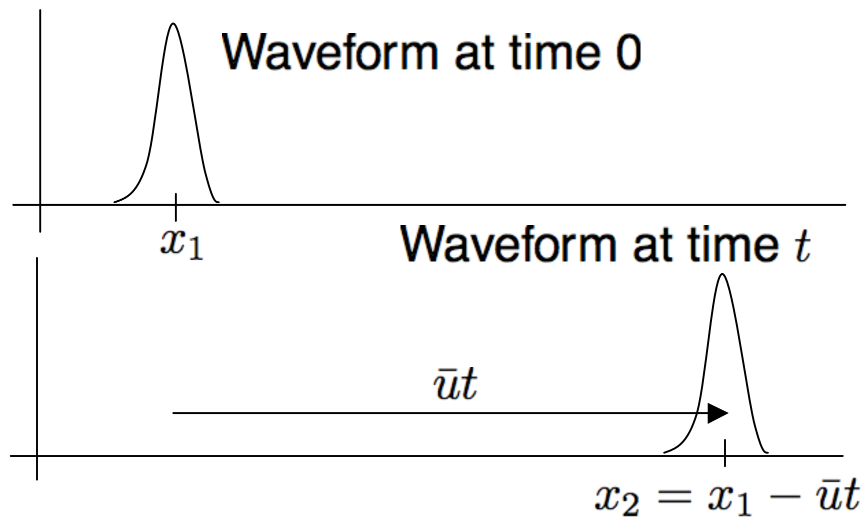
$$\begin{aligned} \dot{y} &= ay \\ y(0) &= y_0 \end{aligned} \quad \Rightarrow y(t) = e^{At} y_0$$

Analytic Solution Techniques: Wave Equation

Wave Equation: Assume that $\bar{u}$ is constant and $D = b = d = 0$

$$\frac{\partial \rho}{\partial t} + \bar{u}\frac{\partial \rho}{\partial x} = 0$$

$$\rho(0, x) = \rho_0(x)$$



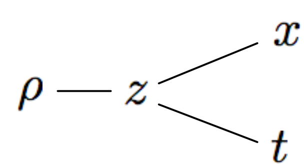
Solution: $\rho(t, x) = \rho_0(x - \bar{u}t)$

Verification: Let $z = x - \bar{u}t$

Thus

$$\rho_t = \frac{d\rho}{dz}\frac{\partial z}{\partial t} = \rho_0'(x - \bar{u}t)(-\bar{u})$$

$$\rho_x = \frac{d\rho}{dz}\frac{\partial z}{\partial x} = \rho_0'(x - \bar{u}t)$$



Hence

$$\rho_t + \bar{u}\rho_x = -\bar{u}\rho_0'(x - \bar{u}t) + \bar{u}\rho_0'(x - \bar{u}t) = 0$$

$$\rho(0, x) = \rho_0(x)$$