MA 573 — PROJECT 5

Due: Monday, October 30

In this project, we are going to model the steady-state temperature distribution for aluminum and copper rods heated by a soldering rod.

1. For our experimental apparatus, we have rectangular, uninsulated, rods with cross-sectional dimensions $a = b = 0.95$ cm and length $L = 70$ cm. The posted .txt files include 15 temperature measurements every 4 cm from 10 cm to 66 cm. A heat source at $x = 0$ provides a fixed, but unknown, heat flux Φ. Derive the model

$$\rho c_\rho \frac{\partial T}{\partial t} = \frac{\partial}{\partial x}\left(k\frac{\partial T}{\partial x}\right) - \frac{2(a+b)h}{ab}\left[T(t,x) - T_{amb}\right] \ , \ 0 < x < L, \tag{1}$$

and boundary conditions

$$k\frac{dT}{dx}(t,0) = \Phi \quad , \quad k\frac{dT}{dx}(t,L) = h[T_{amb} - T(t,L)]. \tag{2}$$

Here T, ρ, c_ρ, k, h and T_{amb} respectively denote the temperature, density, specific heat, thermal conductivity, convective heat transfer coefficient, and ambient room temperature. Initial conditions are specified as

$$T(0,x) = T_0(x). \tag{3}$$

Derive the steady-state model where T_s denotes the steady-state temperature. We will use the steady-state equations in all further computations. Note that we will use the thermal conductivity values $k = 2.37\frac{W}{cm \cdot C}$ and $k = 4.01\frac{W}{cm \cdot C}$ when modeling the aluminum and copper rods.

2. Consider the *insulated* aluminum rod model with the boundary conditions

$$k\frac{dT_s}{dx}(0) = \Phi \ , \ T_s(L) = T_{amb}$$

and parameter $q = \Phi$. Determine the analytic solution and parameter Φ that yields an optimal fit. What do you conclude about the accuracy of this model.

3. Now estimate the parameters $q = [\Phi, h]$ for the *uninsulated* aluminum rod with the boundary conditions (2). Note that the solution is

$$T_s(x;q) = c_1(q)e^{-\gamma x} + c_2(q)e^{\gamma x} + T_{amb} \tag{4}$$

where $\gamma = \sqrt{\frac{2(a+b)h}{abk}}$ and

$$c_1(q) = -\frac{\Phi}{k\gamma}\left[\frac{e^{\gamma L}(h+k\gamma)}{e^{-\gamma L}(h-k\gamma) + e^{\gamma L}(h+k\gamma)}\right] \quad , \quad c_2(q) = \frac{\Phi}{k\gamma} + c_1(q).$$

Explain why you cannot simultaneously estimate the parameter set $q = [\Phi, h, k]$. Plot the optimal fit and data along with the residuals. Do your errors appear to be iid? Are the values for your parameters physically reasonable?

4. For the optimal fit to the aluminum rod data, determine an estimate s^2 for the variance σ_0^2 of the errors and construct the covariance matrix; note that you can do this analytically. Determine 95% confidence intervals for each parameter.

5. Repeat your analysis from 3 and 4 for the copper rod.

6. Finally, estimate the optimal parameters by simultaneously using the data from both the aluminum and copper rods. This will yield one value of Φ and h for both rods. Discuss your results and note any limitations of the model. Can you obtain a better fit if you consider the parameters $q = [\Phi_a, \Phi_c, h]$, where Φ_a and Φ_c denote potentially differing fluxes for the aluminum and copper rods?