

MA 573 — Project 3

Due: Wednesday, October 4

- (1) Here we consider a traffic model where $\rho(t, x)$ is the density of cars per unit length of road, $q(t, x)$ is the rate at which cars flow past x , and u is the speed at which they are moving. For heavy traffic, a car's speed can be modeled with the relation

$$u(\rho) = u_{max}(1 - \rho/M)$$

where u_{max} is the maximum speed in light traffic. What is a physical interpretation of M ?

- (a) Consider the data in Table 1, which is taken from [1] and was collected in the Lincoln Tunnel between New York and New Jersey. Plot the data and determine optimal values of u_{max} and M . How good is the assumption of linearity and how could it be improved?
- (b) Determine a differential equation quantifying the flow of traffic under these conditions.
- (c) As a group, devise and conduct an experiment to obtain similar data on Hillsborough Street. You should obtain at least three sets of values of ρ and u . Can you estimate uncertainty bounds on your data? Be sure to report your experimental conditions.

ρ	0.021	0.037	0.058	0.067	0.103
u	14.3	10.3	7.2	4.9	2.7

Table 1: Density ρ (cars/m) and speeds u (m/s) from [1].

- (2) The Atlantic Ocean off the Outer Banks of North Carolina is often called the “Graveyard of the Atlantic” since over 2000 ships have sunk there since records began in 1526. On August 16, 1918, the British tanker Mirlo struck a mine about 5 miles offshore subsequently sparking a massive explosion. A rescue crew from the Chicamacomico Life Station launched a surfboat and rescued six of the crew.

In this problem, we are going to model the spread of fuel from the wreckage. Since the shore is roughly north-south at this location, you can take it to be the x_2 axis with $x_1 = 0$ denoting the shore and $x_1 = 8000$ m denoting the location of the Mirlo. The ocean currents in this region are tricky since the cold south-flowing Labrador current meets the warm north-flowing Gulf Stream off the coast of North Carolina. We are going to assume the former with the simplified velocity profile

$$\vec{u}(x_1, x_2, t) = \left[0, \frac{-x_1}{1 + x_1/100} \right]. \tag{1}$$

We further assume that fuel leaks into the water at a uniform rate within 15 m of the ship and is zero outside this region; hence we will take the source term to be

$$b(x_1, x_2, t) = \begin{cases} 0.1 & , \quad \sqrt{(x_1 - 8000)^2 + x_2^2} \leq 15 \\ 0 & , \quad \text{otherwise} \end{cases}$$

with units of $\frac{kg}{m^2s}$. You can assume that the ship starts leaking fuel at $t = 0$ and that the fuel has concentration $\rho(x_1, x_2, t)$ with units of kg/m^2 . The diffusivity D can be assumed constant.

- (a) At what rate (kg/s) is fuel being dumped into the water?
- (b) Does the relation (1) for \vec{u} satisfy any conservation properties? Discuss this with regard to steady-state flow. Is this expression reasonable at the shore? How would you need to modify it to incorporate rip-currents flowing out from shore?
- (c) Formulate a PDE quantifying the spread of fuel. Be sure to specify appropriate initial and boundary conditions.

(3) Consider the unforced equation

$$\begin{aligned}\frac{\partial u}{\partial t} &= \alpha \frac{\partial^2 u}{\partial x^2} \\ u(t, 0) &= u(t, 2) = 0 \\ u(0, x) &= \sin(\pi x/2)\end{aligned}\tag{2}$$

with $\alpha = 0.7$. You should determine the analytic solution and then approximate the solution to (2) using central differences in space and backward differences in time. In the same figure, plot the true and approximate solutions at $T = 1$ for appropriate stepsizes to indicate the convergence of the method. You should also compute the maximum absolute error over your space-time grid and report that for a representative set of stepsizes. Is your method achieving the prescribed convergence rate?

(4) Now approximate the solution to (2) using a finite element discretization in space and backward differences in time. Report your solutions and errors in the same manner as in Problem 3.

(5) Finally, approximate the solution of the forced equation

$$\begin{aligned}\frac{\partial u}{\partial t} &= \alpha \frac{\partial^2 u}{\partial x^2} + x(x - 2) \sin(3\pi t) \\ u(t, 0) &= u(t, 2) = 0 \\ u(0, x) &= \sin(\pi x/2),\end{aligned}\tag{3}$$

with $\alpha = 0.7$, using the finite difference technique. Discuss the convergence of the method and indicate techniques that you are using to ascertain convergence (you do not need to compute an analytic solution for this case). When does the solution appear to have reached steady state behavior?

Hint: You can use the `diag` command to create the tridiagonal, Toeplitz matrix for your finite difference and finite element discretizations.

References

- [1] H. Greenberg, "An Analysis of Traffic Flow," *Operations Research*, 7(1), pp. 79-85, 1959.