Math 573 — Exam 1

(1) The goal in this problem is to model the dynamics of an algae population in a long tube filled with water as depicted in Figure 1. The water flowing into the tube at $x = 0$ is pure (no algae) and there is a filter at $x = L$ that prevents algae from leaving the tube. The flow rate is assumed constant and is given by v. The length is sufficiently larger than the radius of a cross-section so you can consider this as a 1-D problem along the center of the tube which is taken to be the x-axis. You should let $\rho(t, x)$ represent the density of the population so the initial density is $\rho_0(x) = \rho(0, x)$.

(a) Construct a model for the algae dynamics that incorporates the following contributions:

(i) c_0 algae eaten by fish per unit time; Is c_0 positive or negative?

(ii) an exponential growth rate of c_1;

(iii) a convective term due to the flow of water;

(iv) diffusion with a rate constant of c_3.

Be sure to motivate the terms in your model.

(b) Determine appropriate boundary conditions for your model.

(c) Determine the solution for the following scenarios. For a fixed point $0 < x < L$, plot the solution as a function of time for each case.

- The exponential growth (ii), convection (iii) and diffusion (iv) are neglected.
- The fish (i), convection (iii) and diffusion (iv) are neglected.
- The fish (i), exponential growth (ii) and diffusion (iv) are neglected.

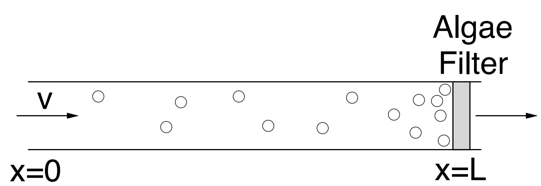

Figure 1: Algae population affected by consumption, growth, convection and diffusion.

(2) Here we are going to model the population of fish released into a stream having velocity $\vec{u}$. Let $\rho(t, x, y, z)$ denote the density of fish (fish/unit volume). At time $t = 0$, a pail full of fish, with initial densition $\rho(0, x, y, z) = \rho_0(x, y, z)$, is released into the river where their movement is governed by the current $\vec{u}$ and diffusion at a rate D.

(a) Derive the continuity equation (conservation of mass) in 3-D and be sure to motivate each step.

(b) Determine an appropriate flux function q to complete your model. You do not need to specify boundary conditions.

(c) Express your model in terms of the material derivative $\frac{D\rho}{Dt}$.

(3) Consider the model

$$\frac{\partial \rho}{\partial t} - D\frac{\partial^2 \rho}{\partial x^2} = \alpha\rho\left(1 - \frac{\rho}{m}\right) - E\rho$$

$$\rho(t,0) = \rho(t,H) = 0$$

$$\rho(0,x) = \rho_0(x).$$

This models the population $\rho(t,x)$ of fish between the shore, $x = 0$, and a distance $x = H$ from the shoreline. Here D is a diffusivity constant, α is the maximum growth rate, m is a carrying density, and $E\rho$ is the harvest rate.

Use central differences in space and forward differences in time to construct a discrete numerical model expressed as a vector-valued system. What issues must be addressed if you replace the forward differencing by backward differencing in time? What would be an advantage to backward differencing?