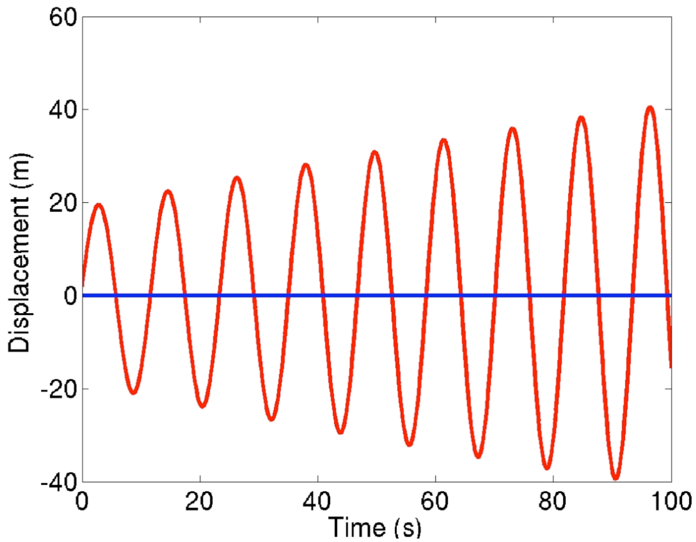
Hypothesis Testing for Model Comparison

"It is the mark of an educated mind to rest satisfied with the degree of precision which the nature of the subject admits and not to seek exactness where only an approximation is possible." Aristotle

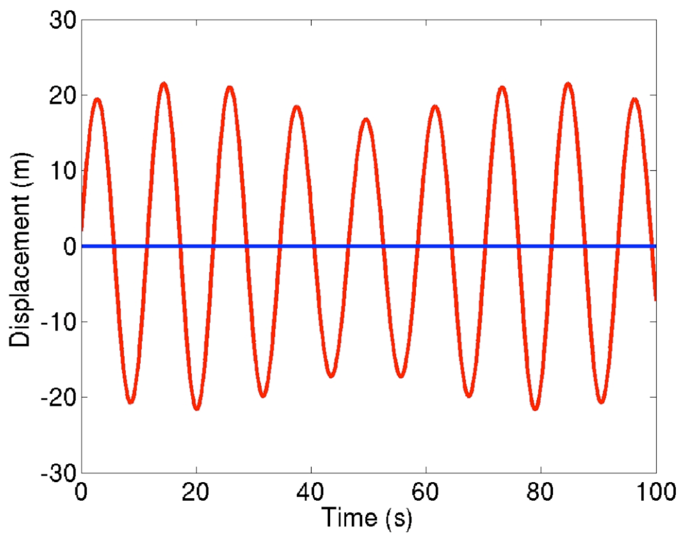
Motivation

Question: Do the following forced spring responses exhibit damping?

Example 1:

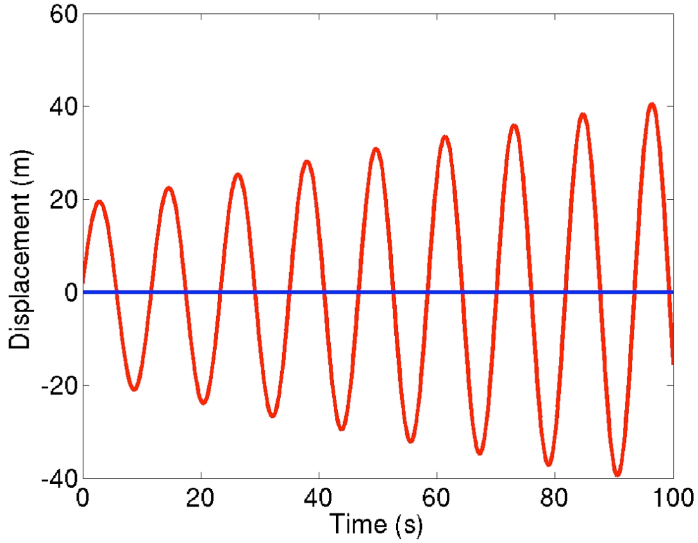


Example 2:



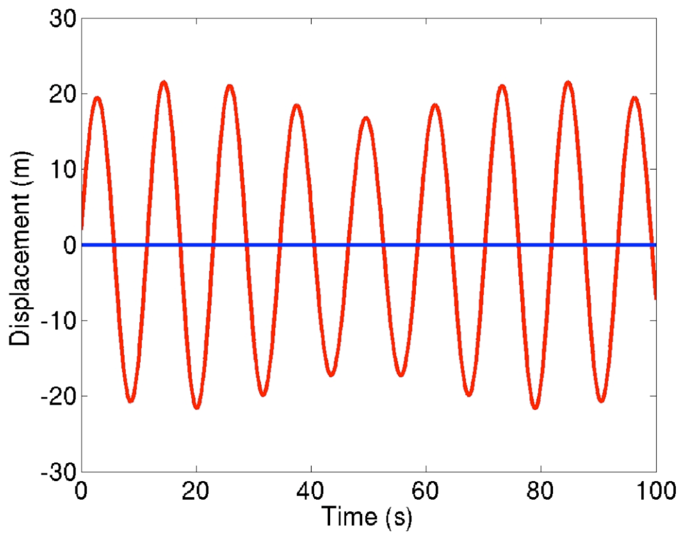
Motivation

Example 1



$$m = 7, k = 2, c = 0.01$$
$$\omega = 0.5445, \omega_0 = 0.5345$$

Example 2:



$$m = 7, k = 2, c = 0$$
$$\omega = 0.6345, \omega_0 = 0.5345$$

Model Comparison

Model: Consider the model

$$m\ddot{y} + c\dot{y} + ky = F_0 \cos(\omega t)$$
$$y(0) = y_0 \ , \ \dot{y}(0) = y_1$$

which has the solution

$$y(t) = e^{(-c/2m)t}[A\cos(\nu t) + B\sin(\nu t)] + \frac{F_0}{\Delta}\cos(\omega t - \delta)$$

where $\omega_0 = \sqrt{k/m}$ and

$$\nu = \frac{\sqrt{4km - c^2}}{2m} \ , \ \Delta = \sqrt{m^2(\omega_0^2 - \omega^2)^2 + c^2\omega^2} \ , \ \delta = \arccos(m(\omega_0^2 - \omega^2)/\Delta)$$

$$A = y_0 - \frac{F_0}{\Delta}\cos\delta \ , \ B = \frac{1}{\nu}\left(y_1 + \frac{cA}{2m} - \frac{F_0\omega}{\Delta}\sin\delta\right)$$

Parameter Spaces:

$$Q_0 = \{q = (m,c,k) \,|\, m > 0, c = 0, k > 0\}$$
$$Q_1 = \{q = (m,c,k) \,|\, m > 0, c > 0, k > 0\}$$

Hypothesis Testing

Hypotheses:

$$H_0 : c = 0$$

$$H_1 : c \neq 0$$

Test Statistic: Let $\bar{q}_0, \bar{q}_1$ denote minimizers over Q_0, Q_1. Take

$$J(q) = \sum_{j=1}^{n} [y_j - y(t_j; q)]^2$$

and consider

$$U_n = n \frac{J(\bar{q}_0) - J(\bar{q}_1)}{J(\bar{q}_1)}$$

Note: $U_n \rightarrow U$ where U is a random variable having a $\chi^2(r)$ distribution

Hypothesis Test:

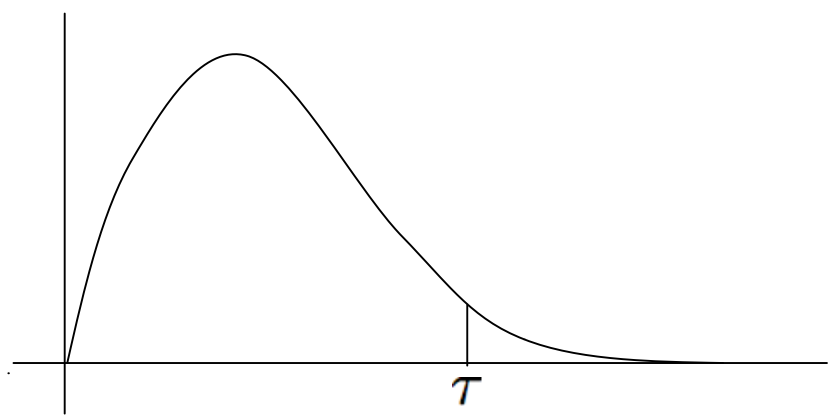
- Choose significance level α and use $\chi^2(r)$ tables to compute threshhold τ so that $P(\chi^2(r) > \tau) = \alpha$

Hypothesis Testing

Hypothesis Test:

- Compute U_n and compare to τ. If $U_n > \tau$, reject H_0 as false. Otherwise, accept.
- r is determined by the dimensionality of the parameter space; $r = 1$ if linear constraint. In general, $Hq = c$ where H is $r \times p$.

  e.g., $H = [0\ ,1\ ,0]$ so $Hq = c = 0$



Hypothesis Testing

Example: [Banks and Fitzpatrick, 1990]

Data Set 1:

$$J(\bar{q}_0) = 180.17$$

$$J(\bar{q}_1) = 106.15$$

$$\Rightarrow U_8 = 8 \cdot \frac{180.17 - 106.15}{106.15} = 5.579$$

Table:

$\chi^2(1)$	
$\alpha = 0.25$	$\tau = 1.32$
$\alpha = 0.10$	$\tau = 2.71$
$\alpha = 0.05$	$\tau = 3.84$
$\alpha = 0.01$	$\tau = 6.63$
$\alpha = 0.005$	$\tau = 10.83$

Conclusion: Reject H_0 at α level of 0.05 or larger