Chapter 9 - Uncertainty Propagation

1. Linear Models

$$\Upsilon_i = XQ + \varepsilon$$

Example 1: Stress-strain model

$$\Upsilon_i = E e_i + E_2 e_i^3 + \varepsilon_i \ , \ i=1,\dots,n$$
$$= f_i(Q) + \varepsilon_i$$

Note: Stress Υ_i (F/A)

Strain $e_i = \lim_{\Delta X \to 0} \frac{\Delta u}{\Delta X} = \frac{\partial u}{\partial X}$

Here
$$Q_1 = E, \ Q_2 = E_2$$

and
$$\begin{bmatrix} \Upsilon_1 \\ \vdots \\ \Upsilon_n \end{bmatrix} = \begin{bmatrix} e_1 & e_1^3 \\ \vdots & \\ e_n & e_n^3 \end{bmatrix} \begin{bmatrix} E \\ E_2 \end{bmatrix} + \begin{bmatrix} \varepsilon_1 \\ \vdots \\ \varepsilon_n \end{bmatrix}$$

Statistics: Take $\bar{q} = [\bar{q}_1, \bar{q}_2] = [\bar{E}, \bar{E}_2]$. Then

$$\mathbb{E}[f_i(Q)] = \bar{E} e_i + \bar{E}_2 e_i^3$$

$$\text{var}[f_i(Q)] = \mathbb{E}[f_i^2(Q)] - \mathbb{E}^2[f_i(Q)]$$
$$= \mathbb{E}[(E e_i + E_2 e_i^3)^2] - (\bar{E} e_i + \bar{E}_2 e_i^3)^2$$
$$= \mathbb{E}[E^2 e_i^2 + 2 E E_2 e_i^4 + E_2^2 e_i^6$$
$$- \bar{E}^2 e_i^2 - 2\overline{E E_2} e_i^4 - \bar{E}_2^2 e_i^6]$$
$$= e_i^2 \text{var}(E) + e_i^6 \text{var}(E_2) + 2 e_i^4 \text{cov}(E, E_2)$$

Note:
$$\mathbb{E}\left(\sum_{i=1}^n a_i X_i\right) = \sum_{i=1}^n a_i \mathbb{E}(X_i)$$
$$\text{var}\left(\sum_{i=1}^n a_i X_i\right) = \sum_{i=1}^n a_i^2 \text{var}(X_i) + 2\sum_{i<j} a_i a_j \text{cov}(X_i, X_j)$$

- See Chapter 4

Example 2: Linear Multiple Regression

$$\Upsilon_i = Q_1 + \sum_{j=2}^p x_{ij} Q_j + \varepsilon_i$$

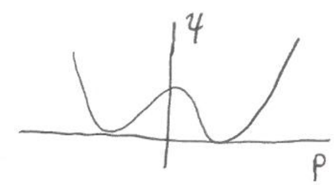
$$\Rightarrow \begin{bmatrix} \Upsilon_1 \\ \vdots \\ \Upsilon_n \end{bmatrix} = \begin{bmatrix} 1 & x_{12} & \cdots & x_{1p} \\ \vdots & & & \\ 1 & x_{n2} & \cdots & x_{np} \end{bmatrix} \begin{bmatrix} Q_1 \\ \vdots \\ Q_p \end{bmatrix} + \begin{bmatrix} \varepsilon_1 \\ \vdots \\ \varepsilon_n \end{bmatrix}$$

Then
$$\mathbb{E}[f_i(Q)] = \bar{q}_1 + \sum_{j=2}^p x_{ij} \bar{q}_j$$
$$\text{var}[f_i(Q)] = \text{var}(Q_1) + \sum_{j=2}^p [x_{ij}^2 \text{var}(Q_j) + 2x_{ij} \text{cov}(Q_1, Q_j)]$$
$$+ \sum_{1<j<k} x_{ij} x_{ik} \text{cov}(Q_j, Q_k)$$

for $i = 1, \dots, n$.

e.g. Height-weight data
$$f(Q) = Q_1 + Q_2 \left(\frac{x}{12}\right) + Q_3 \left(\frac{x}{12}\right)^2$$

e.g. $\Psi = \alpha_1 P^2 + \alpha_{11} P^4 + \alpha_{111} P^6$



Bayesian Analysis: Data of Table 7.1

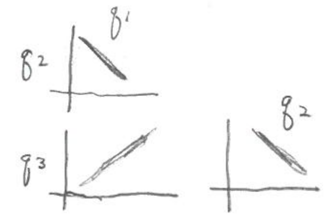
$$\bar{q} = [260.65, -87.74, 11.92]^T$$

$$V = \begin{bmatrix} var(Q_1) & cov(Q_1, Q_2) & cov(Q, Q_3) \\ & & \\ & & \end{bmatrix}$$

$$= \begin{bmatrix} 778.35 & -287.93 & 26.51 \\ -287.93 & 106.61 & -9.82 \\ 26.51 & -9.82 & 0.91 \end{bmatrix}$$

Note: Results from Bayesian inference differ somewhat from OLS results on page 138. Sensitive to MCMC!

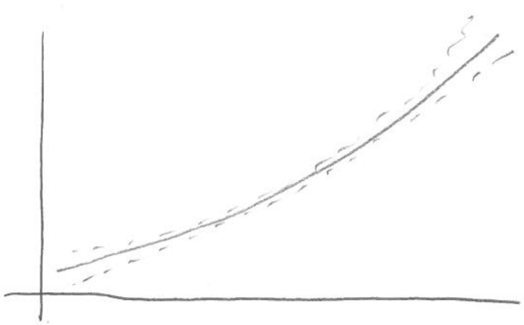
Note: Very correlated and V has small eigenvalue,

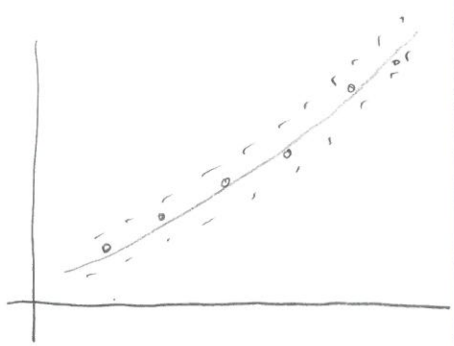


Now,

$$\mathbb{E}[f(Q)] = \bar{q}_1 + \left(\frac{x}{12}\right)\bar{q}_2 + \left(\frac{x}{12}\right)^2 \bar{q}_3$$

$$var[f(Q)] = var(Q_1) + \left(\frac{x}{12}\right)^2 var(Q_2) + \left(\frac{x}{12}\right)^4 var(Q_3)$$
$$+ 2\left(\frac{x}{12}\right) cov(Q_1, Q_2) + 2\left(\frac{x}{12}\right)^2 cov(Q_1, Q_3)$$
$$+ 2\left(\frac{x}{12}\right)^3 cov(Q_2, Q_3)$$





Sampling-Based: Sample out of $\rho_Q(q)$ and $\rho_\varepsilon(\varepsilon)$.

Perturbation Methods: Take

$$Q = \bar{q} + \delta Q \quad \text{Realization - often } 1\sigma$$
$$= [\bar{q}_1 + \delta Q_1, \cdots, \bar{q}_p + \delta Q_p]$$

Then

$$f(Q) = f(\bar{q}) + \sum_{i=1}^{p} \frac{\partial f}{\partial Q_i}\Big|_{\bar{q}} \delta Q_i + H.O.T$$
$$= \bar{y} + \sum_{i=1}^{p} S_i \delta Q_i .$$

Consider $Q = [Q_1, \cdots, Q_p]$ with joint pdf $\rho_Q(q)$. Then

$$E(Q_i) = \bar{q}_i$$

$$var(Q_i) = \int_{\mathbb{R}^p} (q_i - \bar{q}_i)^2 \rho_Q(q) dq$$
$$= \int_{\mathbb{R}^p} (\delta q_i)^2 \rho_Q(q) dq$$

$$cov(Q_i, Q_j) = \int_{\mathbb{R}^p} (q_i - \bar{q}_i)(q_j - \bar{q}_j) \rho_Q(q) dq$$
$$= \int_{\mathbb{R}^p} \delta q_i \delta q_j dq$$

Note: $\mathbb{E}[f(Q)] = \bar{y} \int_{\mathbb{R}^p} \rho_Q(q) dq + \sum_{i=1}^{p} S_i \int_{\mathbb{R}^p} (q_i - \bar{q}_i) \rho_Q(q) dq$
$$= \bar{y}$$

since $\int_{\mathbb{R}^p} (q_i - \bar{q}_i) \rho_Q(q) dq = \bar{q}_i - \bar{q}_i = 0$