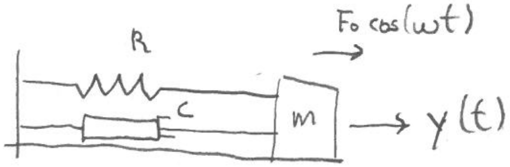
Three Last Models:



Model: $m \dfrac{d^2y}{dt^2} + c\dfrac{dy}{dt} + ky = F_0 \cos \omega t$

$$y(0) = y_0 ,\quad \frac{dy}{dt}(0) = y_1$$

Solution: $y(t) = c_1 e^{r_1 t} + c_2 e^{r_2 t}$

$$+ \frac{F_0}{\sqrt{m^2(\omega_0^2 - \omega^2) + c^2\omega^2}\cos(\omega t - \delta)}$$

$$\left\{ \omega_0^2 = \frac{k}{m} \right.$$

Special Case: $y(0) = 1$

$$\frac{dy}{dt}(0) = c = F_0 = 0$$

Then $y(t) = \cos\left(\sqrt{\frac{k}{m}}\, t\right)$

Here

$$q = (k, m)$$

General: $q = (k, c, m, y_0, y_1)$

Question: Which are most uncertain?

Note: Take $z_1(t) = y(t)$ and $z_2(t) = \frac{dy}{dt}(t)$ to get

$$\frac{d}{dt}\begin{bmatrix} z_1(t) \\ z_2(t) \end{bmatrix} = \begin{bmatrix} 0 & -1 \\ -\frac{k}{m} & -\frac{c}{m} \end{bmatrix}\begin{bmatrix} z_1(t) \\ z_2(t) \end{bmatrix} + \begin{bmatrix} 0 \\ \frac{F_0 \cos(\omega t)}{m} \end{bmatrix}$$

$$\Rightarrow \frac{dz}{dt} = Az(t) + F(t)$$

$$z(0) = z_0$$

Model 2: Scalar exponential process

$$\left.\begin{aligned} \frac{dz}{dt} &= az + b(t) \\ z(0) &= z_0 \end{aligned}\right\} \text{Deterministic}$$

Inputs: $q = [a, z_0, b(t)]$

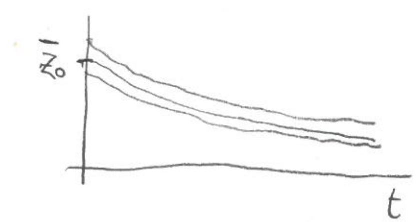
Deterministic Solution:

$$z(t, q) = e^{at}\left[z_0 + \int_0^t e^{-as}\, b(s)\right] ds$$

Random Differential Equation:

$$\frac{dz}{dt} = a(\omega) z(t) + b(t, \omega)$$

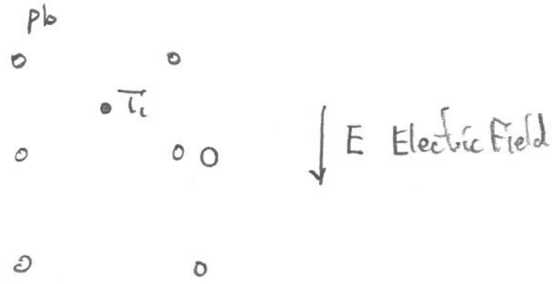
$$z(0) = z_0(\omega)$$

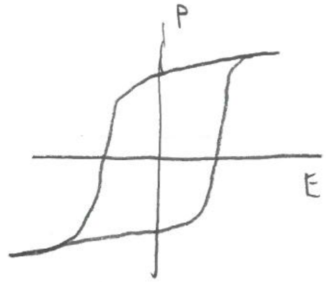


$$\Rightarrow z(t, \omega) = e^{a(\omega)t}\left[z_0(\omega) + \int_0^t e^{-a(\omega)} b(s, \omega)\, ds\right]$$

[Random field]

Model 3: PZT (Lead Zirconate Titanate)

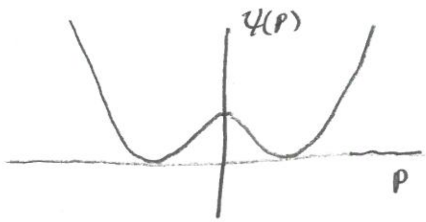




Gibbs Energy: $G(E,P) = \psi(P) - EP$

Helmholtz Energy: $\psi(P) = \alpha_1 P^2 + \alpha_{11} P^4 + \alpha_{111} P^6$

$$\alpha_1 < 0 \, , \; \alpha_{11} > 0$$



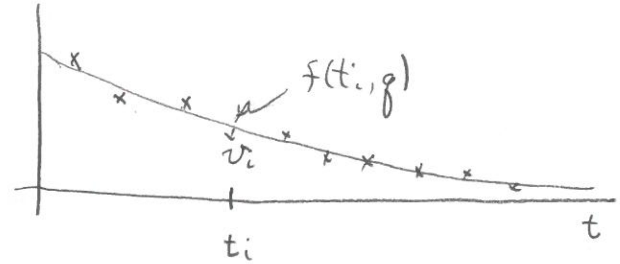
Note: Take $q = [\alpha_1, \alpha_{11}, \alpha_{111}]$ to be random variables

Mathematical Model: Describes physical or biological process

Statistical Model: Describes observation process

Example: Take $f(t,q) = z(t,q) = e^{at}\left[z_0 + \int_0^t e^{-as} b(s)\,ds\right]$

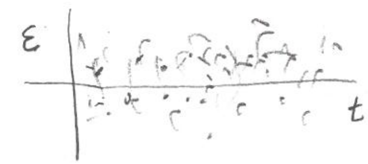
- $f(t_i,q)$: Model predictions at times t_i
- v_i : Measured data with errors ε_i



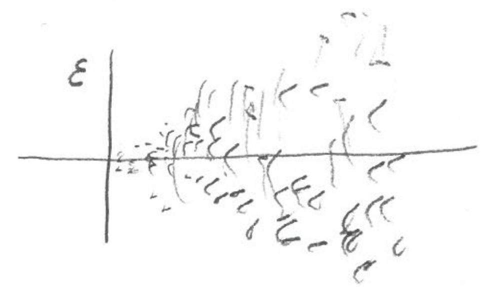
Common Assumptions:

1) ε_i are independent and identically distributed (iid)

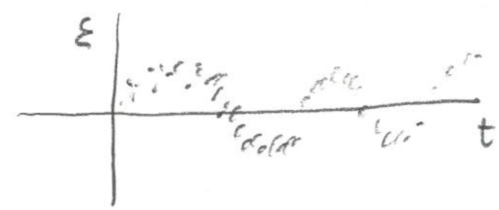
e.g.



iid



Independent but not identically distributed



Not independent

2) $\varepsilon_i \sim N(0,\sigma^2)$

3) $\varepsilon_i \sim P(\lambda)$

Note: Motivates probability and statistics!!

Common Distributions: Read Chapter 4

1) Normal (Gaussian)

$$X \sim N(\mu, \sigma^2) \qquad \text{Two parameters: } \mu, \sigma^2$$

$$f(x) = \frac{1}{\sigma\sqrt{2\pi}} e^{-(x-\mu)^2/2\sigma^2}$$

2) $X \sim \mathcal{U}(-1,1)$ Uniform

$$f(x) = \frac{1}{2}$$

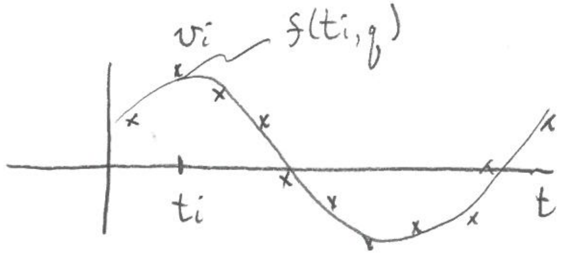
3) Chi-Squared

$$X \sim N(0,1)$$

$$Y = X^2 \sim \chi^2(1)$$

1 degree of freedom

Motivation:



Ordinary Least Squares: Find $q \in Q$ that minimizes

$$J(q) = \sum_{i=1}^{N} [v_i - f(t_i,q)]^2$$

$$\Rightarrow q_{min} = \underset{q}{\arg\min}\, J(q)$$

Note:

$$\min_q J(q) = \min_q \sum_{i=1}^{n} [\varepsilon_i]^2 \quad , \quad \varepsilon_i \sim N(0,\sigma^2)$$

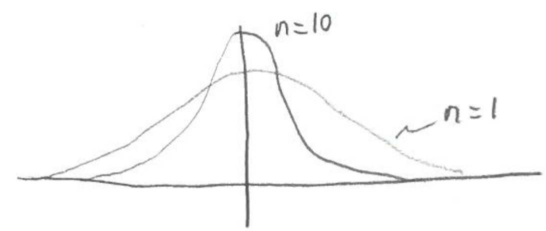
4) Student's T-Distribution

Take $X \sim N(0,1)$ and $Y = X^2 \sim \chi^2(1)$. Suppose Y_i are independent $\chi^2(1)$ and $Z = \sum_{i=1}^{n} Y_i \sim \chi^2(n)$. Then

$$T = \frac{X}{\sqrt{Z/n}}$$

has a student's T-distribution w/ n dof.

Note: Used to construct confidence intervals.



Multiple Random Variables:

e.g., Polarization model $\psi(P) = \alpha_1 P^2 + \alpha_{11} P^4 + \alpha_{111} P^6$

$$q = [\alpha_1, \alpha_{11}, \alpha_{111}]$$

Note: Let $X = [X, Y]$ be bivariate random variable with joint pdf $f(x,y)$.

Marginal: $f_X(x) = \int_{\mathbb{R}} f(x,y)\,dy$

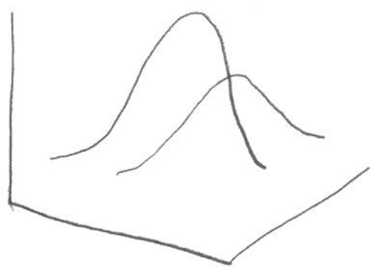
$$f_Y(y) = \int_{\mathbb{R}} f(x,y)\,dx$$

Then X, Y independent if $f(x,y) = f_X(x)\, f_Y(y)$

Conditional Density:

$$f(x|y) = \begin{cases} \dfrac{f(x,y)}{f_Y(y)} & , \ f_X(x) > 0 \\ 0 & , \ \text{else} \end{cases}$$

Marginal



Conditional

Covariance of X, Y: e.g., height + weight

$$\mathrm{cov}(X,Y) = \mathbb{E}[(X - \mathbb{E}(X))(Y - \mathbb{E}(Y))]$$
$$= \mathbb{E}[XY] - \mathbb{E}(X)\mathbb{E}(Y)$$

Pearson (Linear) Correlation:

$$\rho_{XY} = \frac{\mathrm{cov}(X,Y)}{\sigma_X \ \underbrace{\sigma_Y}_{\sqrt{\text{var of } Y}}}$$

Multivariate Normal: $X \sim MVN(\mu, V)$, $X = [X_1, \dots, X_p]$

$$f(X) = \frac{1}{\sqrt{(2\pi)^p \det(V)}} \exp\left[-\frac{1}{2}(X-\mu) V^{-1} (X-\mu)^T\right]$$

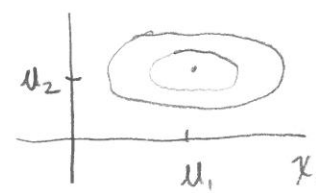
where

$$\mu = [\mu_1, \dots, \mu_p]$$

$$V = \mathrm{cov}(X) = \begin{bmatrix} \mathrm{var}(X_1) & \mathrm{cov}(X_1, X_2) & \cdots & \mathrm{cov}(X_1, X_p) \\ \mathrm{cov}(X_2, X_1) & \mathrm{var}(X_2) & & \\ \vdots & & & \\ \mathrm{cov}(X_p, X_1) & \cdots & & \mathrm{var}(X_p) \end{bmatrix}$$

Note: Symmetric, positive definite

e.g., $p = 2$, $\mathrm{cov}(X_1, X_2) = 0$



Note: X, Y independent $\Rightarrow$ X, Y uncorrelated

Gaussian has $\Leftrightarrow$

Definition: An estimate (real-values) is a rule or procedure for determining attributes of a quantity based on data.

Definition: An estimator is associated random variable or random vector.

e.g.,) Consider $X_1, \ldots, X_n$. **Goal:** estimate mean & variance

Estimators: $\bar{X} = \frac{1}{n} \sum_{i=1}^{n} X_i$ (R.V.) sample mean

$$S^2 = \frac{1}{n-1} \sum_{i=1}^{n} (X_i - \bar{X})^2 \quad \text{(R.V.) sample variance}$$

Estimates: $\bar{x} = \frac{1}{n} \sum_{i=1}^{n} x_i$, s^2 similar

Distributions for the Estimators:

Suppose $X_i \sim N(\mu, \sigma^2)$. Sampling distributions –

$$\bar{X} \sim N\left(\mu, \frac{\sigma^2}{n}\right)$$

$$S^2 \sim \frac{\sigma^2}{n-1} \chi^2(n-1)$$

Interval Estimators:

- Exam scores
- Temperatures (Boiling & freezing)

q_L $\qquad$ q_R

Goal: Determine functions $q_L(x)$ and $q_R(x)$ that bound the location of q,

$$q_L(x) < q < q_R(x),$$

based on realizations $x = [x_1, \ldots, x_n]$ of a random sample $X = [X_1, \ldots, X_n]$.

Interval Estimator: Random interval $[q_L(X), q_R(X)]$

Confidence Interval: Interval estimator plus confidence coef.

- $(1-\alpha) \times 100\%$ Confidence Interval $= [q_L(X), q_R(X)]$ such that

$$P[q_L(X) \le q \le q_R(X)] = 1 - \alpha$$

Example: Suppose $X_i \sim N(\mu, \sigma^2)$ where μ is unknown and σ^2 is known.

Example 4.32

Consider
$$\bar{X} = \frac{1}{n}\sum_{i=1}^{n} X_i.$$

Note:
$$\bar{X} \sim N\left(\mu, \frac{\sigma^2}{n}\right) \Rightarrow \frac{\bar{X}-\mu}{\sigma/\sqrt{n}} \sim N(0,1)$$

Then
$$P\left(-2 < \frac{\bar{X}-\mu}{\sigma/\sqrt{n}} < 2\right) = 0.9545$$

$$\Rightarrow P\left(\bar{X} - \frac{2\sigma}{\sqrt{n}} < \mu < \bar{X} + \frac{2\sigma}{\sqrt{n}}\right) = 0.9545$$

Interval Estimator: $\left[\bar{X} - \frac{2\sigma}{\sqrt{n}}, \bar{X} + \frac{2\sigma}{\sqrt{n}}\right]$

Note: 95.45% Confidence Interval: $\left[\bar{x} - \frac{2\sigma}{\sqrt{n}}, \bar{x} + \frac{2\sigma}{\sqrt{n}}\right]$

where $\bar{x} = \frac{1}{n}\sum_{i=1}^{n} x_i$ ← Realization

Example 4.33: μ, σ^2 both unknown – Need for linear regression

Note:
$$X = \frac{\sqrt{n}(\bar{X}-\mu)}{\sigma} \sim N(0,1)$$

$$Z = \frac{(n-1)S^2}{\sigma^2} \sim \chi^2(n-1)$$

Then
$$T = \frac{X}{\sqrt{Z/(n-1)}} = \frac{\sqrt{n}(\bar{X}-\mu)}{S}$$

has t-distribution with $n-1$ d.o.f.

Goal: Find a & b such that
$$P\left(a < \frac{\sqrt{n}(\bar{X}-\mu)}{S} < b\right) = 1-\alpha$$

Symmetry: Take $b = -a$

Notation: $t_{n-1,1-\frac{\alpha}{2}} \Rightarrow n-1$ dof, prob $1-\frac{\alpha}{2}$

Then $a = t_{n-1,1-\frac{\alpha}{2}}$ so
$$P\left(\bar{X} - \frac{t_{n-1,1-\frac{\alpha}{2}} S}{\sqrt{n}} < \mu < \bar{X} + \frac{t_{n-1,1-\alpha/2} S}{\sqrt{n}}\right) = 1-\alpha$$

Interval: Employ realizations
$$\left[\bar{x} - \frac{t_{n-1,1-\alpha/2}\, s}{\sqrt{n}}, \bar{x} + \frac{t_{n-1,1-\alpha/2}\, s}{\sqrt{n}}\right].$$

Example: Consider n height measurements x_i from population with

$\mu = 67$

$\sigma = 2.5$

Note: height_example.m

Assumption: $X_i \sim N(\mu, \sigma^2)$

MATLAB: >> $X_i = \mu + \sigma * \text{randn}(1,n)$

Recall: $\bar{X} = \frac{1}{n}\sum_{i=1}^{n} X_i$

$$S^2 = \frac{1}{n-1}\sum_{i=1}^{n}(X_i - \bar{X})^2$$

Then

$$\bar{X} \sim N(\mu, \frac{\sigma^2}{n})$$

$$S^2 \sim \frac{\sigma^2}{n-1}\chi^2(n-1)$$

Code: Try with $n = 44$, 400, 4000 and show decrease in variance

Question: How do we plot samples x_i?

1.) Histogram: histogram(X, nbin)
2.) Normalized histogram: histnorm(X)] – Binning is an issue – Demonstrate
3.) Kernel density estimate (kde) • Pages 75-76
   • ksdensity.m MATLAB Statistics toolbox
   • kde.m, kde2d.m MATLAB central

Note: histnorm(X, nbins, 'plot')

Ordinary Least Squares and Likelihood Estimators: Section 4.3

Recall Statistical Model:

$$Y_i = f(t_i, q) + \mathcal{E}_i \quad , \quad i = 1, \dots, n$$

Y_i: Random observations with realizations y_i

$\mathcal{E}_i$: Random observation errors w/ realizations ϵ_i

q: Frequentist theory: True but unknown parameters; not r.v.
   Bayesian: Random variables w/ distributions

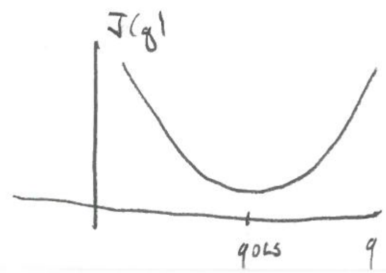
Note: $q \in Q$ admissible parameter space

OLS Estimator and Estimate:

$$\hat{q}_{OLS} = \underset{q \in Q}{\text{argmin}} \sum_{i=1}^{n} [Y_i - f(t_i, q)]^2$$

$$q_{OLS} = \underset{q \in Q}{\text{argmin}} \sum_{i=1}^{n} [y_i - f(t_i, q)]^2$$

Typical Assumption: $\mathcal{E}_i$ iid with true but unknown variance σ^2 and $\mathbb{E}(\mathcal{E}_i) = 0$.



Likelihood Function: page 83

Define $f_r(v; q)$ as parameter-dependent joint density

Note: $r = [r_1, \dots, r_n]$ random vector

$q = [q_1, \dots, q_p]$ unknown parameter, Q admissible space

Define $L: Q \to [0, \infty)$ by

$$L_r(q) = L(q \mid v) = f(v; q)$$

where

q varies over Q

v values are fixed.

Example: Binomial distribution with probability of success q

$$f_r(v; q, n) = P(r = v \mid n, q)$$

$$= \binom{n}{v} q^v (1-q)^{n-v} \quad \text{Discrete}$$

Notes: i) Quantifies probability of obtaining exactly $v = 0, 1, \dots, n$ in n experiments

ii) q, n are known and v is unknown

(viii)

Likelihood: $L(q \mid v, n) = \binom{n}{v} q^n (1-q)^{n-v}$ Continuous

Definition: For n iid rv, due to independence

$$L(q \mid v) = \prod_{i=1}^{n} f(v_i; q)$$

$$= f(v_1; q) \dots f(v_n; q)$$

Assumptions: $\varepsilon_i \overset{iid}{\sim} N(0, \sigma^2) \Rightarrow r_i \sim N(f(t_i, q), \sigma^2)$

Then parameters q, σ^2

$$L(q, \sigma^2 \mid v) = \prod_{i=1}^{n} \frac{1}{\sigma\sqrt{2\pi}} e^{-[v_i - f(t_i, q)]^2 / 2\sigma^2}$$

$$= \frac{1}{(2\pi\sigma^2)^{n/2}} \exp\left[-\sum_{i=1}^{n} [v_i - f(t_i, q)]^2 / 2\sigma^2\right]$$

Maximum Likelihood Estimate: For q, σ^2

$$[q, \sigma^2]_{MLE} = \underset{\substack{q \in Q \\ \sigma^2 \in (0, \infty)}}{\operatorname{argmax}} L(q, \sigma^2 \mid v)$$

Log-Likelihood:

$$\ell(q, \sigma^2 | v) = \frac{n}{2} \ln(2\pi) - \frac{n}{2} \ln(\sigma^2) - \frac{1}{2\sigma^2} \sum_{i=1}^{n} \left(v_i - f(t_i, q) \right)^2$$

Note: $\nabla \ell(q, \sigma^2 | v) = 0$ implies that

$$\sum_{i=1}^{n} \left[v_i - f(t_i, q) \right] \nabla f(t_i, q) = 0$$