Math 540: Project 1

Due Tuesday, February 5

1. Consider the covariance function $C(x, y) = \min(x, y)$ on the interval $[0, 1]$ along with the mean function $\bar{\alpha}(x) = \frac{1}{8}(x + 1)^2$. Compute the first $N = 3$ eigenvalues and eigenfunctions and plot the eigenfunctions. Now compute and plot 1000 realizations of the Karhunen-Loeve expansion truncated at $N = 3$ with random variables $Q_n \sim N(0, 1)$. You can compute the random variables using the command `randn.m`.

For your 1000 realizations, use `histnorm.m` to plot a histogram scaled to unity of the realizations of $\beta(\bar{x}, \omega) = \sum_{n=1}^{3} \sqrt{\lambda_n}\phi_n(\bar{x})Q_n(\omega)$ at $\bar{x} = 0.5$. On the same figure, plot a kernel density estimate (KDE) of the distribution and a normal approximation to it. Is the distribution unbiased and consistent with your choice of $\sigma^2 = 1$? Finally, repeat this plot with $N = 6$ to see if your expansion has converged.

Note: Use the property that

$$\text{var}\left(\sum_{i=1}^{n} a_i X_i\right) = \sum_{i=1}^{n} a_i^2 \text{var}(X_i)$$

2. For the same mean function $\bar{\alpha}(x) = \frac{1}{8}(x + 1)^2$, now consider the exponential covariance function $C(x, y) = e^{-|x-y|/2L}$ on the interval $[-1, 1]$ with the correlation length $L = 5$. The eigenvalues are

$$\lambda_n = \begin{cases} \frac{2L}{1+L^2 w_n^2} & , \text{ if } n \text{ is even,} \\ \frac{2L}{1+L^2 v_n^2} & , \text{ if } n \text{ is odd,} \end{cases}$$

and the eigenfunctions are

$$\phi_n(x) = \begin{cases} \frac{\sin(w_n x)}{\sqrt{1 - \frac{\sin(2w_n)}{2w_n}}} & , \text{ if } n \text{ is even,} \\ \frac{\cos(v_n x)}{\sqrt{1 + \frac{\sin(2v_n)}{2v_n}}} & , \text{ if } n \text{ is odd.} \end{cases}$$

Here w_n and v_n are solutions of the transcendental equations

$$\begin{cases} Lw + \tan(w) = 0 & , \text{ for even } n, \\ 1 - Lv\tan(v) = 0 & , \text{ for odd } n. \end{cases}$$

Compute the first odd and even roots v and w. You can get good initial guesses by plotting the functions and zooming to approximate the roots. You can obtain accurate values using the MATLAB command `fzero.m`. Compute the first two eigenvalues and eigenfunctions and plot the eigenfunctions. Now compute and plot 1000 realizations of the Karhunen-Loeve expansion truncated at $N = 2$ with random variables $Q_n \sim N(0, 1)$.

3. Consider again the covariance function $C(x, y) = \min(x, y)$ on the interval $[0, 1]$ and a Karhunen-Loeve expansion truncated at $N = 3$. You can ignore the contributions from the random variables Q_n. Your objective is to approximate the mean $\bar{\alpha}(x)$ using the representation $\bar{\alpha}(x, q) = q_0 + q_1 x + q_2 x^2$,

where $q = [q_0, q_1, q_2]$ are parameters to be estimated using the data in `KL_data.txt`. In the file, the first column is the x-values at the 21 points $x_j = 0.05(j-1)$ and the second column is values of

$$\begin{aligned} y_j &= \alpha(x_j, q) + \varepsilon_j \\ &= \bar{\alpha}(x_j, q) + \sum_{n=1}^{3} \sqrt{\lambda_n}\phi_n(x_j) + \varepsilon_j \end{aligned}$$

where ε_j is added measurement noise.

You should use optimization software to solve the minimization problem

$$q_{opt} = \arg\min_q \sum_{j=1}^{21} \left[y_j - \alpha(x_j, q)\right]^2 .$$

Plot your approximation to $\alpha(x, q)$ along with the data.

You can see an example regarding the use of `fminsearch.m` in Example 7.16 at the website `https://rsmith.math.ncsu.edu/UQ_TIA/CHAPTER7/index_chapter7.html`. We will revisit this problem again, to estimate the variances σ_n for the random variables Q_n, once we have discussed frequentist and Bayesian inference.