

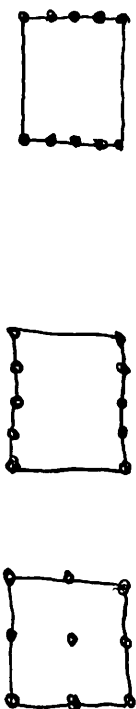
Example: Consider

$$I^{(2)} f = \int_0^1 \int_0^1 \sin(\pi x) \sin(\pi y) dx dy = \left(\frac{2}{\pi}\right)^2 \approx \underline{0.405}$$

Here $f(x, y) = g(x)h(y)$ where $g(x) = \sin(\pi x)$ and $h(y) = \sin(\pi y)$.

Now

$$\begin{aligned} \mathcal{Q}_3^{(2)} f &= (\Delta_1^{(1)} \otimes \Delta_1^{(1)}) f && \begin{array}{|c|} \hline \square \\ \hline \end{array} \\ &+ (\Delta_2^{(1)} \otimes \Delta_1^{(1)}) f && \begin{array}{|c|} \hline \square \\ \hline \square \\ \hline \end{array} \\ &+ (\Delta_1^{(1)} \otimes \Delta_2^{(1)}) f && \begin{array}{|c|} \hline \square \\ \hline \square \\ \hline \end{array} \\ &+ (\Delta_1^{(1)} \otimes \Delta_3^{(1)}) f + (\Delta_3^{(1)} \otimes \Delta_1^{(1)}) f + (\Delta_2^{(1)} \otimes \Delta_2^{(1)}) f \end{aligned}$$



Recall the 1-D Formulas:

$$\begin{aligned} \Delta_2^{(1)} g &= -\frac{1}{4} g(0) + \frac{1}{2} g\left(\frac{1}{2}\right) - \frac{1}{4} g\left(\frac{1}{4}\right) \\ &= \frac{1}{2} \sin\left(\frac{\pi}{2}\right) \\ &= \frac{1}{2} \end{aligned}$$

$$\begin{aligned} \Delta_3^{(1)} g &= \frac{1}{8} \sin(0) + \frac{1}{4} \sin\left(\frac{\pi}{4}\right) + \frac{1}{4} \sin\left(\frac{\pi}{2}\right) + \frac{1}{4} \sin\left(\frac{3\pi}{4}\right) + \frac{1}{8} \sin(\pi) \\ &- \frac{1}{4} \sin(0) && -\frac{1}{2} \sin\left(\frac{\pi}{2}\right) && -\frac{1}{4} \sin(\pi) \\ &= \frac{1}{4} \sin\left(\frac{\pi}{4}\right) - \frac{1}{4} \sin\left(\frac{\pi}{2}\right) + \frac{1}{4} \sin\left(\frac{3\pi}{4}\right) = \frac{1}{4} [\sqrt{2} - 1] \end{aligned}$$

Note:

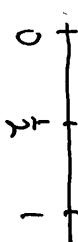
$$l=1 \quad \theta_1^{(1)} = \{0, 1\}$$

$$w_1 = \left[\frac{1}{2}, \frac{1}{2}\right]$$



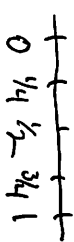
$$l=2 \quad \theta_2^{(1)} = \{0, \frac{1}{2}, 1\}$$

$$w_1 = \left[\frac{1}{4}, \frac{1}{2}, \frac{1}{4}\right]$$



$$l=3 \quad \theta_3^{(1)} = \left\{0, \frac{1}{4}, \frac{1}{2}, \frac{3}{4}, 1\right\}$$

$$w = \left[\frac{1}{8}, \frac{1}{4}, \frac{1}{4}, \frac{1}{4}, \frac{1}{8}\right]$$



Note: The only non-zero component of $\mathcal{Q}_3^{(2)} f$ is

$$\begin{aligned} (\Delta_2^{(1)} \otimes \Delta_2^{(1)}) f &= \left[-\frac{1}{4} g(0) + \frac{1}{2} g\left(\frac{1}{2}\right) - \frac{1}{4} g(1)\right] \left[-\frac{1}{4} h(0) + \frac{1}{2} h\left(\frac{1}{2}\right) - \frac{1}{4} h(1)\right] \\ &+ \left[-\frac{1}{4} g(0) + \frac{1}{2} g\left(\frac{1}{2}\right) - \frac{1}{4} g(1)\right] \left[\frac{1}{2} h\left(\frac{1}{2}\right)\right] \\ &+ \left[-\frac{1}{4} g(0) + \frac{1}{2} g\left(\frac{1}{2}\right) - \frac{1}{4} g(1)\right] \left[-\frac{1}{4} h(0)\right] \\ &= \frac{1}{4} g\left(\frac{1}{2}\right) h\left(\frac{1}{2}\right) \\ &= \frac{1}{4} \sin^2\left(\frac{\pi}{2}\right) \\ &= \underline{0.25} \end{aligned}$$

Now consider

$$Q_4^{(2)} \mathcal{F} = \underbrace{Q_3^{(2)} \mathcal{F} + (\Delta_1^{(1)} \otimes \Delta_4^{(1)}) \mathcal{F} + (\Delta_4^{(1)} \otimes \Delta_1^{(1)}) \mathcal{F}}_U + (\Delta_2^{(1)} \otimes \Delta_3^{(1)}) \mathcal{F} + (\Delta_3^{(1)} \otimes \Delta_2^{(1)}) \mathcal{F}.$$

Here

$$\begin{aligned} (\Delta_2^{(1)} \otimes \Delta_3^{(1)}) \mathcal{F} &= \Delta_2^{(1)} g \cdot \left[\frac{1}{4} \sin\left(\frac{\pi}{4}\right) \right] \\ &+ \Delta_2^{(1)} g \cdot \left[-\frac{1}{4} \sin\left(\frac{\pi}{2}\right) \right] \\ &+ \Delta_2^{(1)} g \cdot \left[\frac{1}{4} \sin\left(\frac{3\pi}{4}\right) \right] \\ &= \frac{1}{2} \left[\frac{1}{4} \cdot \frac{\sqrt{2}}{2} - \frac{1}{4} + \frac{1}{4} \cdot \frac{\sqrt{2}}{2} \right] \\ &= \frac{1}{8} [\sqrt{2} - 1] \end{aligned}$$

and

$$\begin{aligned} (\Delta_3^{(1)} \otimes \Delta_2^{(1)}) \mathcal{F} &= \Delta_3^{(1)} g \left[-\frac{1}{4} \sin(0) + \frac{1}{2} \sin\left(\frac{\pi}{2}\right) - \frac{1}{4} \sin(\pi) \right] \\ &= \frac{1}{2} \cdot \frac{1}{4} [\sqrt{2} - 1] \\ &= \frac{1}{8} [\sqrt{2} - 1] \end{aligned}$$

Thus

$$Q_4 \mathcal{F} = \frac{1}{4} + \frac{1}{4} [\sqrt{2} - 1] \approx .3536$$

