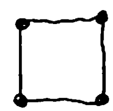
Example: Consider

$$I^{(2)} f = \int_0^1 \int_0^1 \sin(\pi x)\sin(\pi y)\,dx\,dy = \left(\frac{2}{\pi}\right)^2 \simeq \underline{0.405}$$

Here $f(x,y) = g(x)h(y)$ where $g(x) = \sin(\pi x)$ and $h(y) = \sin(\pi y)$.

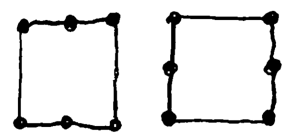
Now

$$Q_3^{(2)} f = (\Delta_1^{(1)} \otimes \Delta_1^{(1)}) f$$

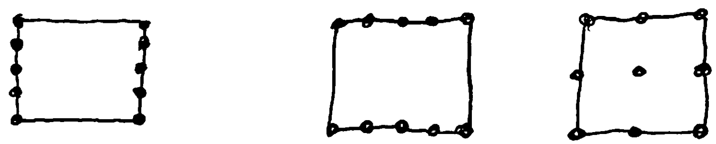


$$+ (\Delta_2^{(1)} \otimes \Delta_1^{(1)}) f$$

$$+ (\Delta_1^{(1)} \otimes \Delta_2^{(1)}) f$$



$$+ (\Delta_1^{(1)} \otimes \Delta_3^{(1)}) f + (\Delta_3^{(1)} \otimes \Delta_1^{(1)}) f + (\Delta_2^{(1)} \otimes \Delta_2^{(1)}) f$$



Recall the 1-D Formulas:

$$\Delta_2^{(1)} g = -\frac{1}{4} g(0) + \frac{1}{2} g\left(\frac{1}{2}\right) - \frac{1}{4} g\left(\frac{1}{4}\right)$$

$$= \frac{1}{2}\sin\left(\frac{\pi}{2}\right)$$

$$= \frac{1}{2}$$

$$\Delta_3^{(1)} g = \frac{1}{8}\sin(0) + \frac{1}{4}\sin\left(\frac{\pi}{4}\right) + \frac{1}{4}\sin\left(\frac{\pi}{2}\right) + \frac{1}{4}\sin\left(\frac{3\pi}{4}\right) + \frac{1}{8}\sin(1)$$

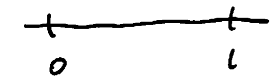
$$- \frac{1}{4}\sin(0) \qquad - \frac{1}{2}\sin\left(\frac{\pi}{2}\right) \qquad - \frac{1}{4}\sin(1)$$

$$= \frac{1}{4}\sin\left(\frac{\pi}{4}\right) - \frac{1}{4}\sin\left(\frac{\pi}{2}\right) + \frac{1}{4}\sin\left(\frac{3\pi}{4}\right) = \frac{1}{4}\left[\sqrt{2} - 1\right]$$

Note:

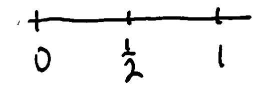
$l=1$ $\quad \Theta_1^{(1)} = \{0, 1\}$

$\omega_1 = \left[\frac{1}{2} \ \frac{1}{2}\right]$



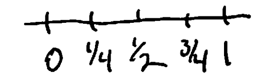
$l=2$ $\quad \Theta_2^{(1)} = \{0, \frac{1}{2}, 1\}$

$\omega_1 = \left[\frac{1}{4}, \frac{1}{2}, \frac{1}{4}\right]$



$l=3$ $\quad \Theta_3^{(1)} = \{0, \frac{1}{4}, \frac{1}{2}, \frac{3}{4}, 1\}$

$\omega = \left[\frac{1}{8}, \frac{1}{4}, \frac{1}{4}, \frac{1}{4}, \frac{1}{8}\right]$



Note: The only non-zero component of $Q_3^{(2)} f$ is

$$(\Delta_2^{(1)} \otimes \Delta_2^{(1)}) f = \left[-\frac{1}{4} g(0) + \frac{1}{2} g\left(\frac{1}{2}\right) - \frac{1}{4} g(1)\right]\left[-\frac{1}{4} h(0)\right]$$

$$+ \left[-\frac{1}{4} g(0) + \frac{1}{2} g\left(\frac{1}{2}\right) - \frac{1}{4} g(1)\right]\left[\frac{1}{2} h\left(\frac{1}{2}\right)\right]$$

$$+ \left[-\frac{1}{4} g(0) + \frac{1}{2} g\left(\frac{1}{2}\right) - \frac{1}{4} g(1)\right]\left[-\frac{1}{4} h(0)\right]$$

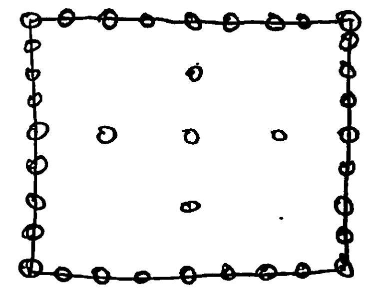
$$= \frac{1}{4} g\left(\frac{1}{2}\right) h\left(\frac{1}{2}\right)$$

$$= \frac{1}{4}\sin^2\left(\frac{\pi}{2}\right)$$

$$= \underline{0.25}$$

Now consider

$$Q_4^{(2)} f = Q_3^{(2)} f + \underbrace{(\Delta_1^{(1)} \otimes \Delta_4^{(1)}) f + (\Delta_4^{(1)} \otimes \Delta_1^{(1)}) f}_{0}$$
$$+ (\Delta_2^{(1)} \otimes \Delta_3^{(1)}) f + (\Delta_3^{(1)} \otimes \Delta_2^{(1)}) f.$$



Here

$$(\Delta_2^{(1)} \otimes \Delta_3^{(1)}) f = \Delta_2^{(1)} g \cdot \left[\tfrac{1}{4} \sin\left(\tfrac{\pi}{4}\right)\right]$$
$$+ \Delta_2^{(1)} g \cdot \left[-\tfrac{1}{4} \sin\left(\tfrac{\pi}{2}\right)\right]$$
$$+ \Delta_2^{(1)} g \cdot \left[\tfrac{1}{4} \sin\left(\tfrac{3\pi}{4}\right)\right]$$
$$= \tfrac{1}{2}\left[\tfrac{1}{4} \cdot \tfrac{\sqrt{2}}{2} - \tfrac{1}{4} + \tfrac{1}{4} \cdot \tfrac{\sqrt{2}}{2}\right]$$
$$= \tfrac{1}{8}\left[\sqrt{2} - 1\right]$$

and

$$(\Delta_3^{(1)} \otimes \Delta_2^{(1)}) f = \Delta_3^{(1)} g \left[-\tfrac{1}{4} \sin(0) + \tfrac{1}{2} \sin\left(\tfrac{\pi}{2}\right) - \tfrac{1}{4} \sin(1)\right]$$
$$= \tfrac{1}{2} \cdot \tfrac{1}{4}\left[\sqrt{2} - 1\right]$$
$$= \tfrac{1}{8}\left[\sqrt{2} - 1\right]$$

Thus

$$Q_4 f = \tfrac{1}{4} + \tfrac{1}{4}\left[\sqrt{2} - 1\right] \approx .3536$$