Global Sensitivity Analysis

Reading: Chapter 15

Example: Portfolio model

$$Y = c_1 Q_1 + c_2 Q_2$$

Take

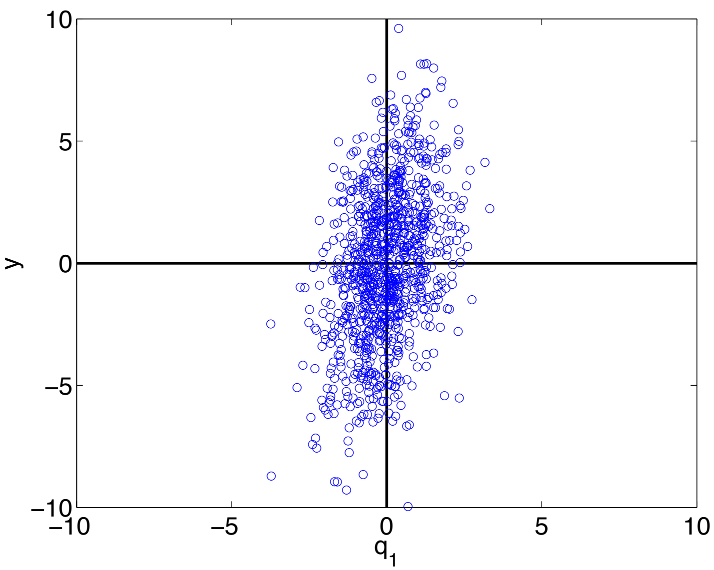
$$c_1 = 2, c_2 = 1$$

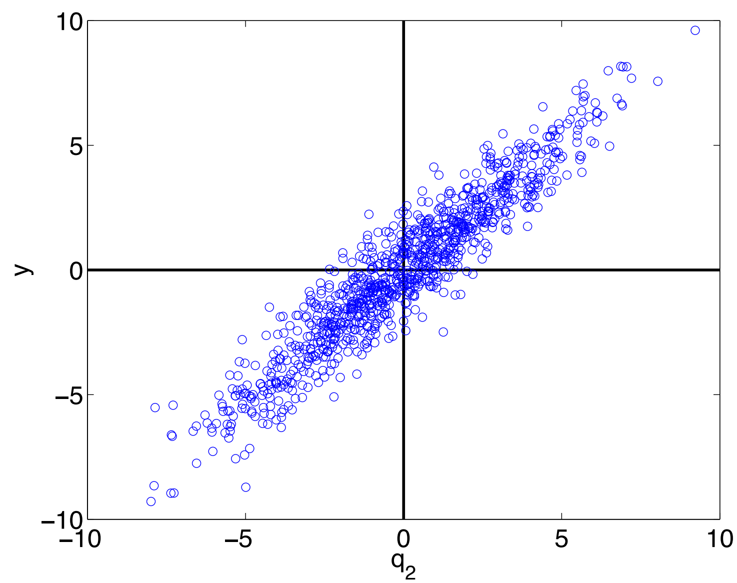
$Q_1 \sim N(0, \sigma_1^2)$ with $\sigma_1 = 1$

$Q_2 \sim N(0, \sigma_2^2)$ with $\sigma_1 = 3$

Note:

- Q_1 and Q_2 represent hedged portfolios
- c_1 and c_2 amounts invested in each portfolio
- $\sigma_Y^2 = c_1^2 \sigma_1^2 + c_2^2 \sigma_2^2 = 13$





Local Sensitivity: $s_i \equiv \dfrac{\partial Y}{\partial Q_i} \Rightarrow s_1 = 2 > s_2 = 1$

Morris Screening

Note: Consider uniformly distributed parameters on $\Gamma = [0,1]^p$

Elementary Effect:

$$d_i^j = \frac{f(q^j + \Delta e_i) - f(q^j)}{\Delta}$$

i^{th} parameter

j^{th} sample

$$\Delta \in \left\{ \frac{1}{\ell - 1}, \cdots, 1 - \frac{1}{\ell - 1} \right\}$$

ℓ is level

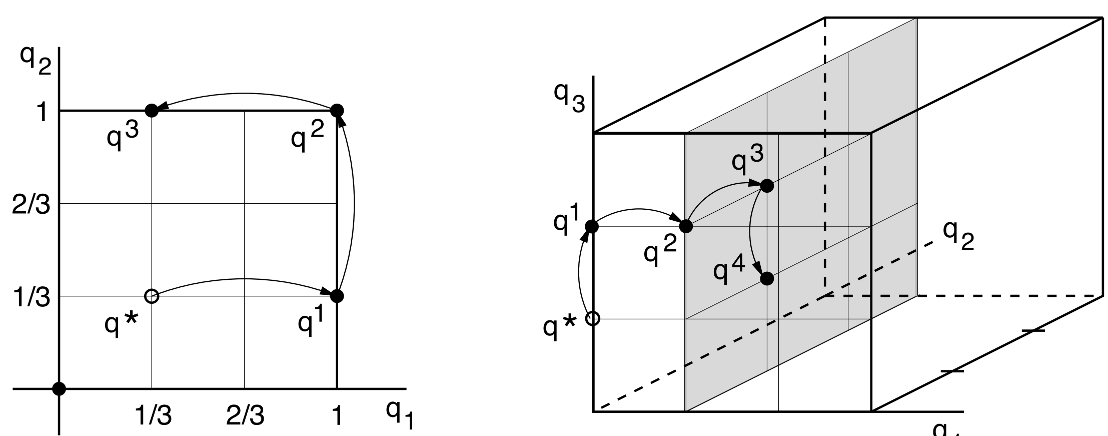
Global Sensitivity Measures: r samples

$$\mu_i^* = \frac{1}{r} \sum_{j=1}^{r} |d_i^j(q)|$$

$$\sigma_i^2 = \frac{1}{r-1} \sum_{j=1}^{r} \left(d_i^j(q) - \mu_i \right)^2 \ , \ \mu_i = \frac{1}{r} \sum_{j=1}^{r} d_i^j(q)$$

Random Differencing:

Note: Slightly different use of q^j



(a) (b)

Morris Screening

Strategy: For each index $j = 1, \cdots, r$, sample a seed values $q^* \in \rho_Q(q)$ and specify $p+1$ parameter values required to approximate p elementary effects using random orientation matrix

$$B^* = [J_{p+1,1}\, q^* + \frac{\Delta}{2}\, [(2B - J_{p+1,p})\, D^* + J_{p+1,p}]\, P^*$$

- D^*: $p \times p$ diagonal matrix with elements in $\{-1, 1\}$
- P^*: Randomly permute columns of $p \times p$ identity
- B: $(p+1) \times p$ strictly lower triangular matrix of ones
- J: $(p+1) \times p$ matrix of ones

Example: $k = 2,\ \ell = 4,\ \Delta = 2/3$. Seed Value: $q^* = [1/3, 1/3]$

$$D^* = \begin{bmatrix} 1 & 0 \\ 0 & -1 \end{bmatrix},\ P^* = \begin{bmatrix} 0 & 1 \\ 1 & 0 \end{bmatrix},\ B = \begin{bmatrix} 0 & 0 \\ 1 & 0 \\ 1 & 1 \end{bmatrix},\ J = \begin{bmatrix} 1 & 1 \\ 1 & 1 \\ 1 & 1 \end{bmatrix}$$

Random Orientation Matrix:

$$B^* = \begin{bmatrix} 1 & 1/3 \\ 1 & 1 \\ 1/3 & 1 \end{bmatrix}$$

Issue: Density choice

Variance-Based Methods

Sobol Representation: For now, take $Q_i \sim \mathcal{U}(0,1)$ and $\Gamma = [0,1]^p$

Take

$$f(q) = f_0 + \sum_{i=1}^{p} f_i(q_i) + \sum_{1 \le i < j \le p} f_{ij}(q_i, q_j)$$

subject to

$$\int_0^1 f_i(q_i) dq_i = \int_0^1 f_{ij}(q_i, q_j) dq_i = \int_0^1 f_{ij}(q_i, q_j) dq_j = 0$$

to ensure

$$\int_\Gamma f_i(q_i) f_j(q_j) dq_i dq_j = \int_\Gamma f_i(q_i) f_{ij}(q_i, q_j) dq_i dq_j = 0$$

Then

Notation: $q_{\sim i} = [q_1, \cdots, q_{i-1}, q_{i+1}, \cdots, q_p]$

$$f_0 = \int_\Gamma f(q) dq$$

$$f_i(q_i) = \int_{\Gamma^{p-1}} f(q) dq_{\sim i} - f_0$$

$$f_{ij}(q_i, q_j) = \int_{\Gamma^{p-2}} f(q) dq_{\sim \{ij\}} - f_i(q_i) - f_j(q_j) - f_0$$

Variance-Based Methods

Notation:

$$\mathbb{E}(Y|q_i) = \int_{\Gamma^{p-1}} f(q)dq_{\sim i}$$

$$\mathbb{E}(Y|q_i, q_j) = \int_{\Gamma^{p-2}} f(q)dq_{\sim\{ij\}}$$

Note:

$$f_0 = \mathbb{E}(Y)$$

$$f_i(q_i) = \mathbb{E}(Y|q_i) - f_0$$

$$f_{ij}(q_i, q_j) = \mathbb{E}(Y|q_i, q_j) - f_i(q_i) - f_j(q_j) - f_0.$$

Total Variance:

$$D = \mathrm{var}(Y) = \int_{\Gamma} f^2(q)dq - f_0^2$$

Partial Variances:

$$D_i = \int_0^1 f_i^2(q_i)dq_i$$

$$D_{ij} = \int_0^1 \int_0^1 f_{ij}^2(q_i, q_j)dq_i dq_j.$$

Sobol Indices:

$$S_i = \frac{D_i}{D} \quad , \; S_{ij} = \frac{D_{ij}}{D} \quad , \; i, j = 1, \cdots, p$$

$$S_{T_i} = S_i + \sum_{j=1}^{p} S_{ij}$$

Variance-Based Methods

Variance Interpretations: Since

$$\mathbb{E}[\mathbb{E}(Y|q_i)] = \int_0^1 \left[\int_{\Gamma^{p-1}} f(q) dq_{\sim i} \right] dq_i = f_0,$$

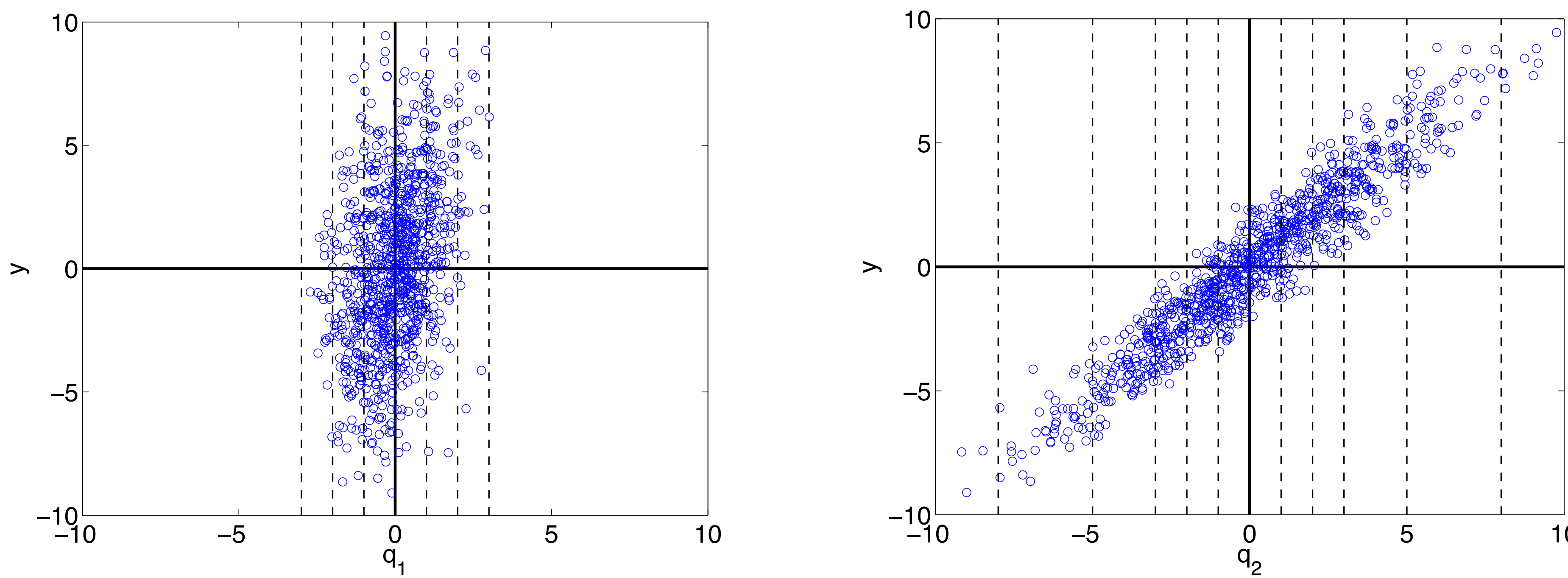
it follows that

$$D_i = \text{var}[\mathbb{E}(Y|q_i)] \Rightarrow S_i = \frac{\text{var}[\mathbb{E}(Y|q_i)]}{\text{var}(Y)}$$

Similarly

$$S_{T_i} = 1 - \frac{\text{var}[\mathbb{E}(Y|q_{\sim i})]}{\text{var}(Y)} = \frac{\mathbb{E}[\text{var}(Y|q_{\sim i})]}{\text{var}(Y)}$$

Note: Implementation algorithm discussed in Section 15.1.3.



Variance-Based Methods

Recall: $\mathrm{var}(f) = \mathbb{E}(f^2) - [\mathbb{E}(f)]^2$

Then

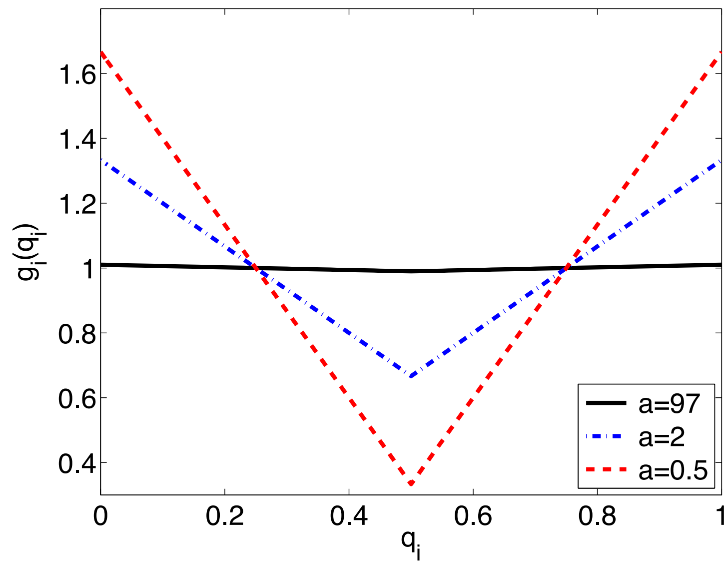
$$\begin{aligned}
D_i = & \int_0^1 f_i^2(q_i)dq_i \\
= & \int_0^1 \left[\int_{\Gamma^{p-1}} f(q)dq_{\sim i} - f_0 \right]^2 dq_i \\
= & \int_0^1 \left[\int_{\Gamma^{p-1}} f(q)dq_{\sim i} \right]^2 dq_i - f_0^2 \\
= & \mathbb{E}\left[\mathbb{E}(Y|q_i)\right]^2 - \left[\mathbb{E}[\mathbb{E}(Y|q_i)]\right]^2 \\
= & \mathrm{var}[\mathbb{E}(Y|q_i)]
\end{aligned}$$

Global Sensitivity Analysis

Example: Sobol function

$$Y = \prod_{i=1}^{p} g_i(Q_i) \quad , \; g_i(Q_i) = \frac{|4Q_i - 2| + a_i}{1 + a_i}$$

where $a_i \geq 0$ are fixed, deterministic coefficients that determine relative importance of parameters.



Note: For $Q_i \sim \mathcal{U}(0,1), i = 1, \cdots, p$.

$$D_i = \text{var}[\mathbb{E}(Y|q_i)] = \frac{1}{3(1+a_i)^2}$$

$$D_{ij} = \text{var}[\mathbb{E}(Y|q_i, q_j)] - D_i - D_j = D_i D_j$$

$$D = \text{var}(Y) = -1 + \prod_{i=1}^{p} (1 + D_i)$$

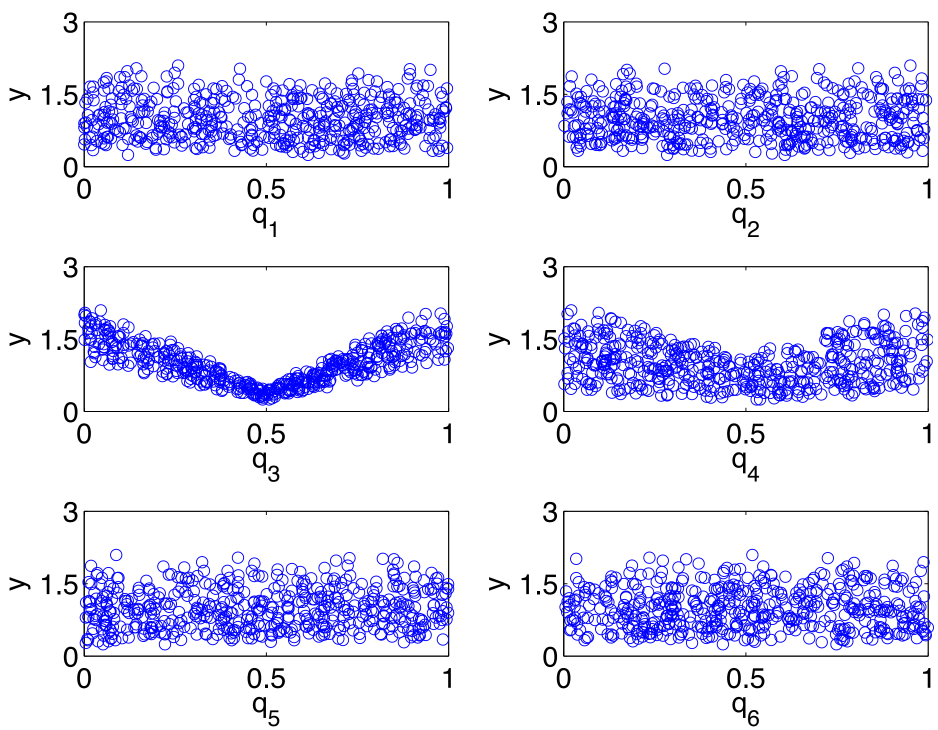
Global Sensitivity Analysis

Sobol Indices:

	Q_1	Q_2	Q_3	Q_4	Q_5	Q_6
a_i	78	12	0.5	2	97	33
D_i	5.3×10^{-5}	2.0×10^{-3}	1.5×10^{-1}	3.7×10^{-2}	3.5×10^{-5}	2.9×10^{-4}
S_i	2.8×10^{-4}	1.0×10^{-2}	7.7×10^{-1}	1.9×10^{-1}	1.8×10^{-4}	1.5×10^{-3}
S_{T_i}	3.3×10^{-4}	1.2×10^{-2}	8.0×10^{-1}	2.2×10^{-1}	2.1×10^{-4}	1.8×10^{-3}

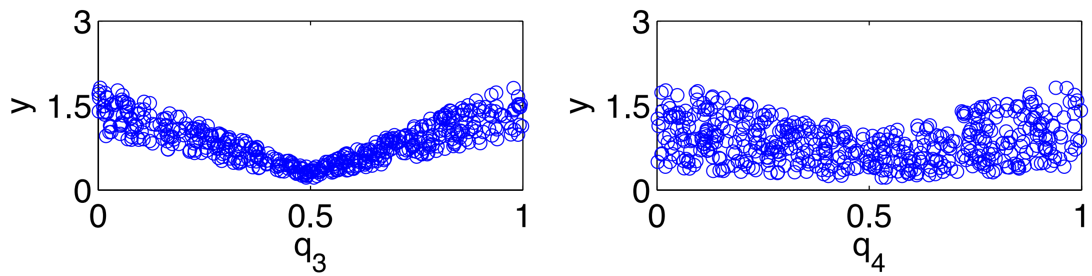
Morris Indices: With $\ell = 4, \Delta = \frac{2}{3}, r = 4$

	Q_1	Q_2	Q_3	Q_4	Q_5	Q_6
μ_i	-0.006	-0.078	-0.130	-0.004	0.012	-0.004
μ_i^*	0.056	0.277	1.760	1.185	0.035	0.099
σ_i	0.064	0.321	2.049	1.370	0.041	0.122



Insensitive Parameters: Take

$$q_1 = q_2 = q_5 = q_6 = \frac{1}{2}$$



Global Sensitivity Analysis

Example: Spring model

$$m\frac{d^2z}{dt^2} + kz = 0$$

$$z(0) = 1 \; , \; \frac{dz}{dt}(0) = 0$$

Responses: For q = [k,m], consider

$$y = f(q) = \cos\left(\sqrt{\frac{k}{m}} \cdot \frac{\pi}{2}\right)$$

$$y = \int_0^{\pi/2} \cos\left(\sqrt{\frac{k}{m}}t\right) dt = \sqrt{\frac{m}{k}}$$

SIR Disease Example

SIR Model:

$$\frac{dS}{dt} = \delta N - \delta S - \gamma k I S \quad , \; S(0) = S_0 \qquad \text{Susceptible}$$

$$\frac{dI}{dt} = \gamma k I S - (r + \delta) I \quad , \; I(0) = I_0 \qquad \text{Infectious}$$

$$\frac{dR}{dt} = rI - \delta R \quad , \; R(0) = R_0 \qquad \text{Recovered}$$

Note: Parameter set $q = [\gamma, k, r, \delta]$ is not identifiable

Assumed Parameter Distribution:

$$\gamma \sim \mathcal{U}(0,1) \; , \; k \sim \mathrm{Beta}(\alpha, \beta) \; , \; r \sim \mathcal{U}(0,1) \; , \; \delta \sim \mathcal{U}(0,1)$$

Infection Coefficient	Interaction Coefficient	Recovery Rate	Birth/death Rate

Response:

$$y = \int_0^5 R(t,q)dt$$

SIR Disease Example

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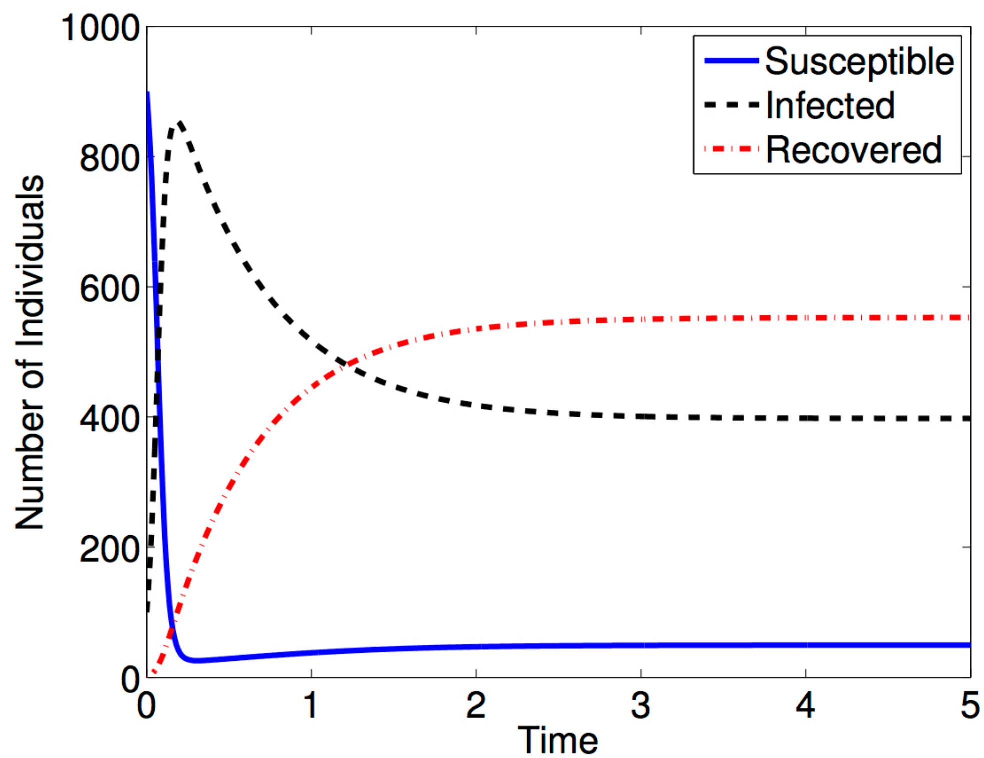
Typical Realization:

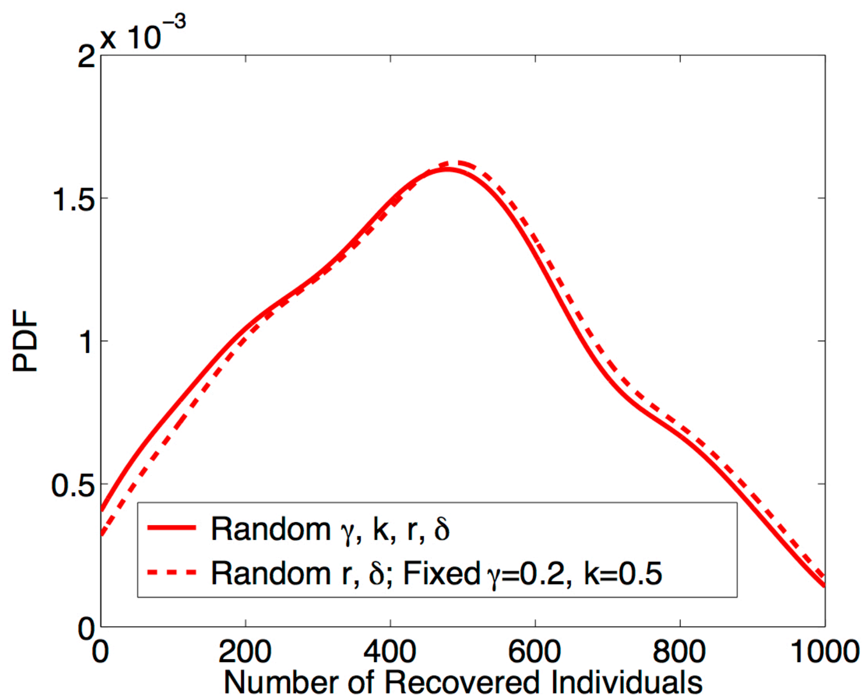


SIR Disease Example

Global Sensitivity Measures:

		γ	k	r	δ
	S_i	0.0997	0.0312	0.7901	0.1750
Sobol	S_{T_i}	-0.0637	-0.0541	0.5634	0.2029
	μ_i^* $(\times 10^3)$	0.2532	0.2812	2.0184	1.2328
Morris	σ_i $(\times 10^3)$	0.9539	1.6245	6.6748	3.9886

Result: Densities for $R(t_f)$ at $t_f = 5$



Note: Can fix non-influential parameters